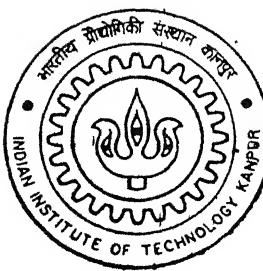


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# **STRUCTURAL DYNAMIC MODELLING AND ANALYSIS OF HINGELESS HELICOPTER ROTOR BLADES**

**By**

**Vaitla Laxman**



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**DEPARTMENT OF AEROSPACE ENGINEERING**

**Indian Institute of Technology Kanpur**

**July, 2002**

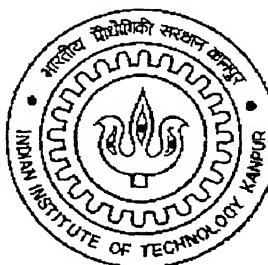
# **STRUCTURAL DYNAMIC MODELING AND ANALYSIS OF HINGELESS HELICOPTER ROTOR BLADES**

*A thesis submitted*  
in partial fulfillment of the requirements  
for the degree of

**Master of Technology**

by

**VAITLA LAXMAN**



to the

**DEPARTMENT OF AEROSPACE ENGINEERING  
INDIAN INSTITUTE OF TECHNOLOGY KANPUR  
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## CERTIFICATE

It is certified that the work contained in the thesis titled “ **STRUCTURAL DYNAMIC MODELING AND ANALYSIS OF HINGELESS HELICOPTER ROTOR BLADES** “, by **VAITLA.LAXMAN** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.



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## Abstract

In this work the formulation of structural dynamic equation of motion for a general helicopter rotor blade with swept tips is attempted. The work aims to solve the problem of the bearingless and the hingeless configuration of the rotor blade. The equations of motion have been derived using Hamilton's principle and the Finite Element Method is applied for its solution. The formulation is validated by comparing the results of the present analysis for a uniform hingeless rotor blade with that of those available in literature. Results have also been generated for practical helicopter rotor blade.

The effect of root offset on the natural frequency and mode shapes of the hingeless rotor blade has also been analysed. The coupling effect of geometric pitch on the natural frequency and mode shapes of the practical rotor blade has also been analysed.

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Vaitla Laxman

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# Nomenclature

$a$	Torque offset
$e_1, e_2$	Root offset
$\hat{e}_x, \hat{e}_y, \hat{e}_z$	Unit vector along X, Y, Z axes
$E_o$	Reference modulus
$Im_{\eta\eta}, Im_{\zeta\zeta}, Im_{\eta\zeta}$	Mass momentum of inertia of the beam cross-section
$[K^g]$	Centrifugal stiffening matrix
$[K^E]$	Linear stiffness matrix
$l$	Length of the blade
$l_e$	Length of each finite element ( $= \frac{l}{N}$ )
$m$	Mass per unit length of blade
$m\eta_m, m\zeta_m$	First moments of the beam cross-section
$[M]$	Mass matrix
$[M^1], [M^2]$	Matrices defined for writing the variation of kinetics
$[M^3], [M^4]$	Energy in matrix form
$[M^c]$	Coriolis damping matrix
$N_b$	Number of blades in the rotor system
$N$	Number of finite elements
$O_H$	Hub center
$\{q\}$	Vector of finite element nodal displacements
$\vec{r}_p$	Position vector of a point 'p' on the deformed $k^{\text{th}}$ blade
$R_x, R_y, R_z$	Components of perturbational hub motion

Tt	Time
T	Kinetic Energy
$[T_y]$	Transformation matrix between orthogonal co-ordinate system i and j
$u_k, v_k, w_k$	$k^{\text{th}}$ blade deformation in axial, lead-lag and flap directions
U	Strain Energy
$V_H$	Hub velocity
$\bar{V}$	Velocity of a point 'p' on $k^{\text{th}}$ blade
$V_x, V_y, V_z$	Components of $\bar{V}$ in X, Y, and Z direction
$\{V^L\}, \{V^{NL}\}, \{V^I\}$	Vectors defined in the expression of kinetic energy variation
$\{V\}, \{W\}, \{\Phi\}, \{U\}$	Vectors of element nodal degrees of freedom
$v'_k$	$= \frac{d(v_k/l)}{d(x/l)}$
$w'_k$	$= \frac{d(w_k/l)}{d(x/l)}$
$W_e$	External work due to nonconservative forces
$x_k$	Coordinate along $k^{\text{th}}$ blade axis
$x, y, z$	Coordinate of a point in the $\hat{e}_x - \hat{e}_y - \hat{e}_z$ system
$x, \eta, \zeta$	Coordinate of a point in the $\hat{e}_x - \hat{e}_n - \hat{e}_\zeta$ system
$\bar{x}$	$= \frac{x}{l}$
$Z_u, Z_v, Z_w, Z_\phi$	Notation used for writing the beam kinetic energy in concise form
$Z'_v, Z'_w, \bar{Z}_u, \bar{Z}_v,$	

$\bar{Z}_w, \bar{Z}_\phi, \bar{Z}'_v, \bar{Z}'_w,$	
$\alpha$	Warping amplitude
$\beta_d$	Blade pre-droop angle
$\beta_k$	Local slope in flap bending of $k^{\text{th}}$ blade
$\beta_p$	Blade pre-cone angle
$\beta_s$	Blade pre-sweep angle
$\overline{\gamma}_v, \overline{\gamma}_\zeta$	Transverse shear strain at the elastic axis
$\zeta_k$	Local slope in lag bending of the $k^{\text{th}}$ blade
$\varepsilon$	Non dimensional parameter representing the order of magnitude of typical elastic blade bending slope
$\epsilon_v, \gamma_v, \gamma_\zeta$	Strain components
$\sigma_v, \sigma_\eta, \sigma_\zeta$	Stress components
$\tau_0$	Initial twist rate of blade
$\theta_G, \theta_g$	Geometric pitch in $k^{\text{th}}$ blade
$\phi_k$	Elastic twist
$\theta_I$	Control pitch input
$\theta_r, \theta_v, \theta_z$	Rigid body perturbational rotation in roll-pitch-yaw
$\Lambda_a$	Tip anhedral angle (positive upwards)
$\Lambda_s$	Tip sweep angle (apt-sweep angle)
$\rho$	Density
$\{\Phi_c\}, \{\Phi_q\}$	Arrays of Hermitic cube and quadratic polynomials respectively
$\{\Phi'_c\}, \{\Phi'_q\}$	First derivative of $\{\Phi_c\}$ and $\{\Phi_q\}$ with respect to $x$
$\psi_k$	Azimuthal angle of $k^{\text{th}}$ blade
$\psi_0$	Non- dimensional time ( $\psi_0 = \Omega_0 t$ )
$\Psi$	Cross-sectional wraping function

$\vec{w}_k$	Angular velocity of the $k^{\text{th}}$ blade
$\Omega$	Speed of rotation of rotor
$\Omega_0$	Constant speed of rotation
$\omega_x, \omega_y, \omega_z$	Components of $\vec{\omega}_k$ in x, y, and z direction
$( )'$	Differentiation of $( )$ with respect to x
$d( )$	Differential of $( )$
$( )_\eta, ( )_\zeta$	Differentiation with respect to $\eta$ and $\zeta$
$( )_x, ( )_{xx}$	Differentiation with respect to x of variables u, v, w and $\phi$
$\delta( )$	Variation of $( )$
$( ), ( )$	$\frac{\partial}{\partial \psi_0}, \frac{\partial}{\partial \psi_0^2}$
{ }	Vector
[ ]	Matrix
1,2,3,4,5,6	Quantities referring to corresponding coordinate system

# Chapter 1

## Introduction

The time varying loads on the main rotor system contributes significantly to the vibration in helicopter. Therefore, the structural dynamic characteristics of the rotor blade and also the dynamic characteristics of the fuselage have a very strong influence on the vibratory levels in helicopter. Any analytical study pertaining to the dynamics of helicopters requires the development of suitable mathematical models for:

- Rotor blade
- Fuselage
- Rotor-fuselage interface

Rotor blade model essentially consists of the development of structural, inertial and aerodynamic operators associated with its motion. The fuselage model is represented by an idealized structural model of a three dimensional structure. The rotor-fuselage interface model must represent both the geometry of the interface as well as the aerodynamic interaction in an appropriate manner. The fundamental tool, necessary for the analysis and design of helicopters, is the rotor blade dynamic model.

During operation, the blades experience large bending and centrifugal loads. In order to relieve the rotor bending moments experienced by the blades, early rotor blades were provided with flap and lag hinges at the root of the blade. In addition, a pitch control bearing was provided to control rotors. Such rotor systems are usually referred to as articulated rotors. A schematic diagram of an articulated rotor system is shown in Fig.1.1. The large number of moving parts leads to a mechanically complex rotor hub system accompanied by the associated wear out problem requiring frequent maintenance and replacement of parts. With advancement in technology, increasing emphasis has been

placed on the development of hingeless rotor systems. The construction of these rotors is relatively simple, because of the absence of flap and lag hinges; but a pitch bearing is still provided for blade pitch control. A schematic diagram of hingeless rotor system is shown in Fig. 1.2. The hub assembly and the main rotor blade geometry of practical helicopter are shown in Fig. 1.3.

## 1.1 Structural Modeling

Since the rotor blades are long, slender beams, they will undergo moderate deformation. A non-linear strain-displacement model is used to describe the coupling effects between axial, bending and torsional deformations. Generally, the strains are assumed to be small in comparison to unity. Such an assumption is consistent with the design requirement based on fatigue life consideration which states the rotor blades must be designed to have an operating strain level well below the elastic limit of the blade material.

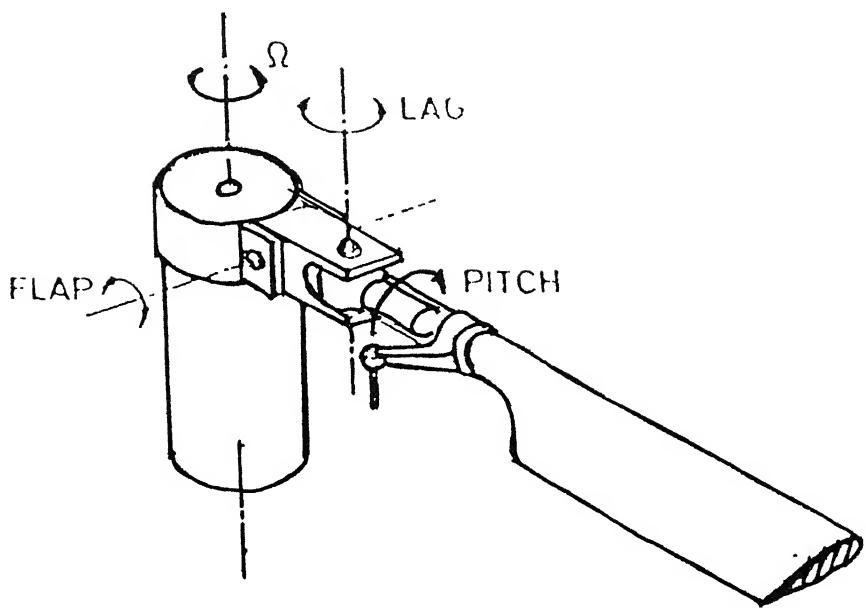
In order to develop an indigenous rotor blade dynamic model, Punit Kumar Gupta [1] formulated a finite element of a hingeless rotor blade under-going flap, lag, torsion and axial deformations. The rotor blade was treated as a straight uniform beam. Subsequently, Venu Gopal [2] extended the model by including non-uniform properties of the blade and swept tip. These rotor models included all the complex geometric parameters such as torque offset, blade root offset, pre-cone angle, pre-droop angle, pre-sweep angle, pre-twist, tip sweep angle, tip anhedral angle and hub motion. However, in these studies rotor speed was taken as constant.

During starting up, the helicopter rotor experiences variable speeds. In order to consider transient conditions in rotor r p.m, it has been treated as a variable.

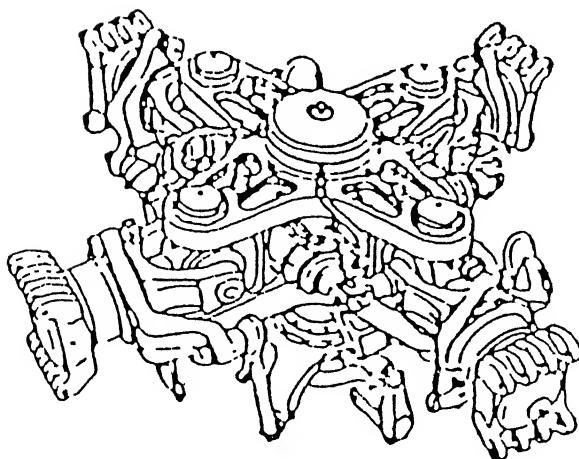
## 1.2 Objectives

Objectives of present study are

- 1 To develop a most general beam type finite element model for helicopter rotor blade, including all the complex geometric parameters such as torque offset, blade root offset, pre-cone angle, pre-droop angle, pre-sweep angle, pre-twist, tip sweep angle, tip anhedral angle and hub motion
- 2 To validate the model by comparing the results of this study with that of those available in literature.
3. To check the model for practical data, provided by industry.
4. To conduct detailed studies on the dynamic behavior of rotor blade to determine the effects of root offset and geometric pitch on natural frequencies and mode shapes



a) Schematic diagram



b) Actual rotor hub

Figure 1.1 Schematic diagram of articulated rotor

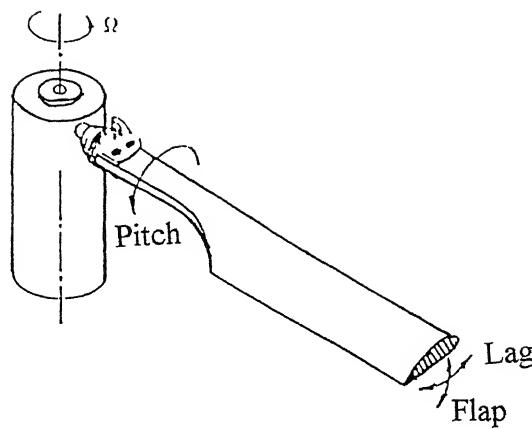
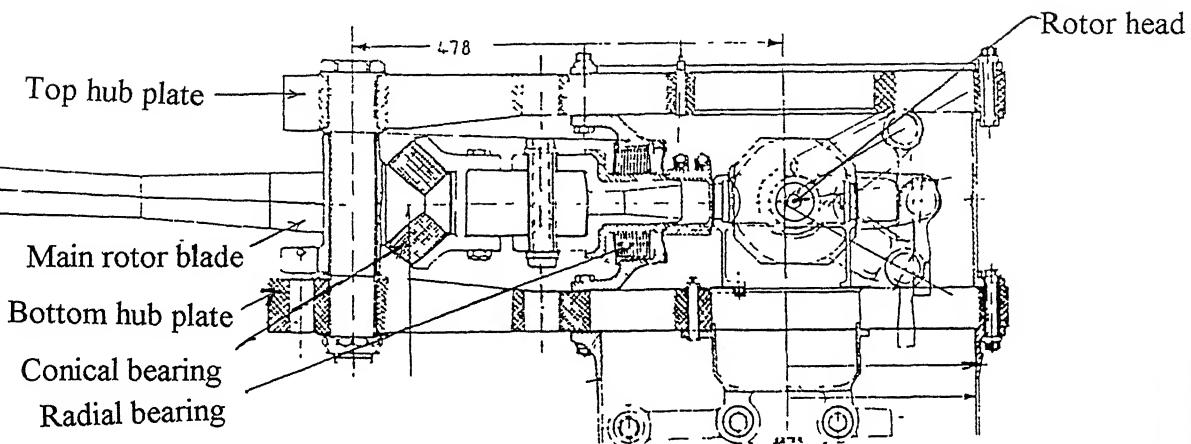
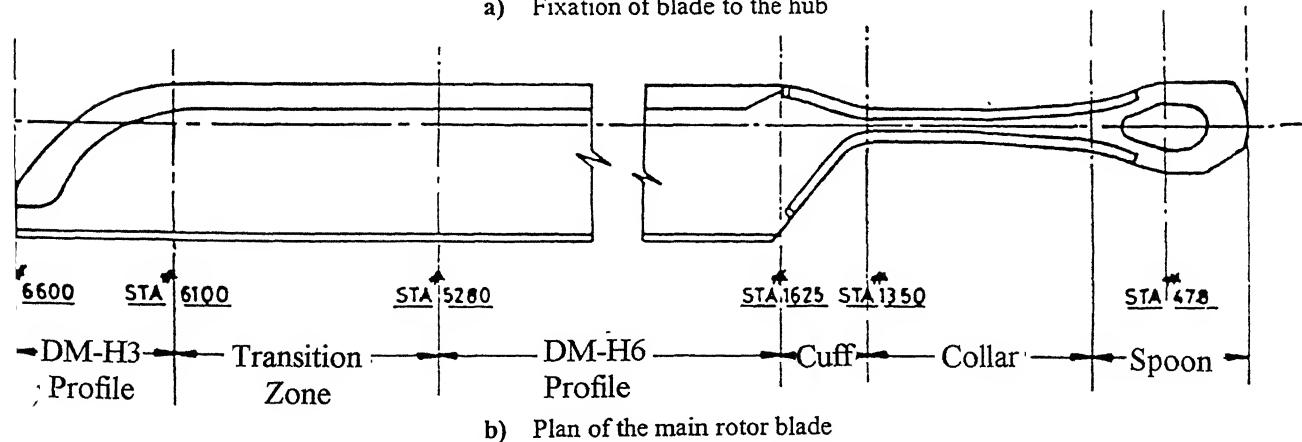


Figure 1.2 Schematic diagram of hingeless rotor

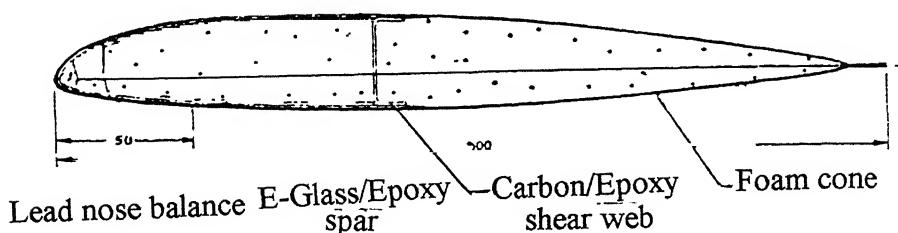


a) Fixation of blade to the hub



b) Plan of the main rotor blade

ST Steel cap      Carbon/Epoxy skin



c) Cross section of main rotor blade

Figure 1.3 Practical hingeless rotor blade

# Chapter 2

## Rotor Blade Model and Assumptions

Helicopter rotor blades are long slender beams attached to the hub through a complex geometrical and mechanical arrangement. The geometrical parameters describing the configuration of the rotor blade-hub system is shown in Fig. 2.1. The parameter  $a$  represents the torque offset, which is the distance from the center of rotation (hub center) to reference axis of the blade.  $e_1$  and  $e_2$  represent blade root offset distances from the hub.  $\beta_p$  represents the pre-cone angle defining the orientation of the blade pitch control axis with respect to the hub plane.  $\beta_s$  and  $\beta_d$  represent the pre-sweep and pre-droop angles, respectively, and representing the inclination of the blade reference axis with respect to the pitch control axis. The blade consists of a straight portion and a swept tip whose orientation relative to the straight portion is described by a sweep angle  $\Lambda_s$  and an anhedral angle  $\Lambda_a$ .

### 2.1 Basic Assumptions

In the formulation of the dynamic model of the rotor blade with swept tip, following assumptions are made:

1. The blade is treated as an elastic beam.
2. The blade is modeled by beam finite elements along the elastic axis of the blade
3. The rotor shaft is rigid
4. The blade cross section has a general shape with distinct shear center, aerodynamic center and center of mass.
5. The blade undergoes moderate deformation in flap, lag, torsion and axial modes.
6. The blade has non-uniform properties along the span though it is made of isotropic material.

## 2.2 Ordering Scheme

In the formulation of the equations of motion of a rotor blade with swept tip undergoing moderate deformations, a large number of higher order terms are generated. In order to identify and eliminate higher order terms in a consistent manner, an ordering scheme is employed. This ordering scheme is based on the assumption that the slopes of the deformed elastic blade are moderate, and of order  $\varepsilon$  ( $0.10 \leq \varepsilon \leq 0.20$ ). Orders of magnitude are then assigned to various non-dimensional physical parameters governing the rotor blade dynamic problem, in terms of  $\varepsilon$ . In the derivation of the governing equations, higher order terms (terms of order greater than  $\varepsilon$ ) are neglected with respect to terms of order 1 term, i.e.,

$$O(1) + O(\varepsilon^3) \approx O(1)$$

The order of magnitude of various non-dimensional parameters governing this problem are given below:

Order 1:

$$\cos \phi_k, \sin \phi_k, \Lambda_a, \Lambda_s = O(1)$$

$$\frac{x_k}{l}, \frac{\Omega}{\Omega_0}, \theta_I = O(1)$$

$$\frac{1}{\Omega_0} \frac{\partial}{\partial t} (\quad) = \frac{\partial}{\partial \phi} (\quad) = O(1)$$

$$l \frac{\partial}{\partial x_k} (\quad) = \frac{\partial}{\partial x'_k} (\quad) = O(1)$$

Order  $\varepsilon^{1/2}$ :

$$\theta_{GK} = O(\varepsilon^{1/2})$$

Order  $\varepsilon$ :

$$\frac{a}{l}, \frac{e_1}{l}, \frac{e_2}{l}, \frac{\eta}{\varsigma}, \frac{v_k}{l}, \frac{w_k}{l} = O(\varepsilon)$$

$$v'_k, w'_k, \phi, \beta_p, \beta_d, \beta_s = O(\varepsilon)$$

Order  $\varepsilon^{3/2}$ :

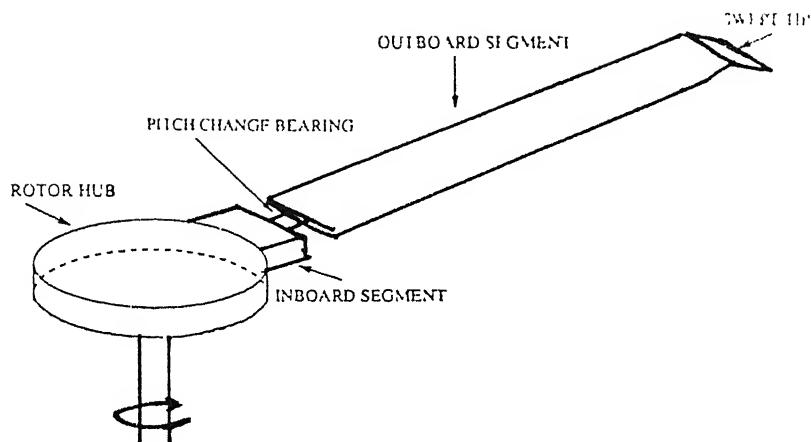
$$\text{Im}_{\eta\eta}, \text{Im}_{\varsigma\varsigma} = O(\varepsilon^{3/2})$$

$$\frac{R_x}{l}, \frac{R_y}{l}, \frac{R_z}{l}, \theta_x, \theta_y, \theta_z = O(\varepsilon^{3/2})$$

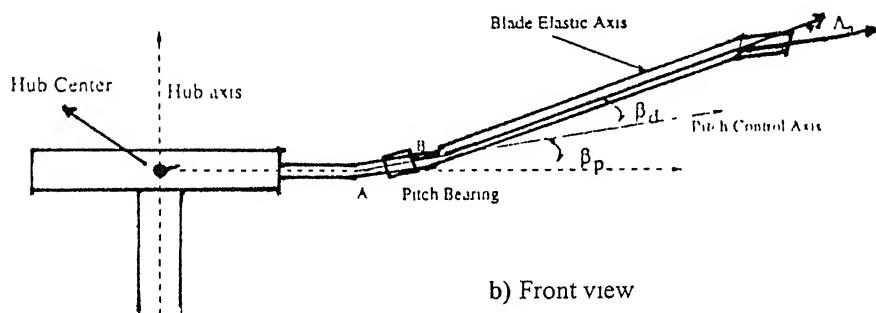
Order  $\varepsilon^2$ :

$$\frac{u_k}{l}, u'_k, m\eta_m, m\varsigma_m = O(\varepsilon^2)$$

It is important to note that ordering schemes are based on physical understanding of the behaviour of actual blade configurations. Hence, care must be exercised in deleting higher order terms, based on this ordering scheme.



a) Schematic model



b) Front view

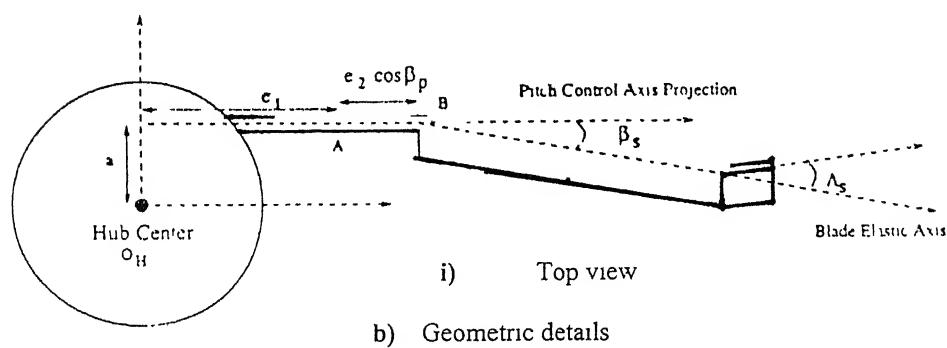


Figure 2.1 Rotor blade with swept tip and anhedral

# Chapter 3

## Coordinate systems

The description of the complex deformation of a rotor requires several coordinate systems. The transformation relation between quantities referred in various inertial, non-inertial coordinate systems is to be established before deriving the equations of motion of the rotor blade. The relation between two orthogonal systems  $X_i, Y_i, Z_i$  and  $X_j, Y_j, Z_j$  with  $\hat{e}_{ix}, \hat{e}_{iy}, \hat{e}_{iz}$  and  $\hat{e}_{jx}, \hat{e}_{jy}, \hat{e}_{jz}$  as unit vectors along the respective axes can be written as

$$\begin{Bmatrix} \hat{e}_{ix} \\ \hat{e}_{iy} \\ \hat{e}_{iz} \end{Bmatrix} = [T_{ij}] \begin{Bmatrix} \hat{e}_{jx} \\ \hat{e}_{jy} \\ \hat{e}_{jz} \end{Bmatrix} \quad (3.1)$$

Where the transformation matrix  $[T_{ij}]$  can be obtained using the Euler angles required to rotate the j-system so as to make it parallel to i-system. The coordinate systems used in deriving the equation of motion for the rotor model are described below:

### 3.1 Hub fixed inertial system – R

The coordinate system ‘R’, shown in Fig. 3.1, has its origin at the center  $O_R$  of the unperturbed hub. The  $x_R$  axis is pointing towards the helicopter tail and  $z_R$  is pointing upwards. The unit vectors along the three axes are  $\hat{e}_{xR}, \hat{e}_{yR}$  and  $\hat{e}_{zR}$ .

### 3.2 Hub fixed moving system – H

The coordinate system H, shown in Fig. 3.2, is a body fixed system with its origin fixed at the center of rotor hub  $O_H$ . Prior to perturbational motion of the hub, H-system coincides with R-system. If  $\theta_x, \theta_y$ , and  $\theta_z$  represents the sequential yaw-pitch-roll rotations, then the transformation matrix  $[T_{RH}]$  can be written as

$$[T_{HR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_r & \sin\theta_r \\ 0 & -\sin\theta_r & \cos\theta_r \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.2)$$

Since  $\theta_r$ ,  $\theta_y$  and  $\theta_z$  are assumed to be of order  $\varepsilon^{\frac{3}{2}}$ , sine and cosine functions can be approximated as:

$$\sin\theta \approx \theta ; \cos\theta \approx 1$$

Substituting the approximation,  $[T_{HR}]$  can be simplified as:

$$[T_{HR}] = \begin{bmatrix} 1 & \theta_z & -\theta_y \\ \theta_x\theta_y - \theta_z & 1 & \theta \\ \theta_x\theta_z + \theta_y & \theta_y\theta_z - \theta_x & 1 \end{bmatrix} \quad (3.3)$$

### 3.3 Hub fixed rotating system – 1

The coordinate system 1, shown in Fig. 3.3, rotates about  $z_{1H}$  axis with variable speed  $\Omega_0$  of the rotor. Its origin is fixed at hub center  $O_H$ . This system can be obtained by rotating H system by an azimuthal angle  $\psi_k$  of the  $k^{th}$  blade about  $z_{1H}$  axis. The transformation matrix is given as:

$$[T_{1H}] = \begin{bmatrix} \cos\psi_k & \sin\psi_k & 0 \\ -\sin\psi_k & \cos\psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.4)$$

Where,  $\psi_k$  is azimuthal angle of  $k^{th}$  blade

$$\psi_k = \psi_0 + (k-1)\frac{2\pi}{N_b} \text{ and } \psi_0 = \Omega_0 t$$

### 3.4 Rotating system – 2K

The coordinate system 2K, shown in Fig. 3.4, is a blade fixed coordinate system, which rotates with  $k^{th}$  blade. The origin of the 2K system is at the location of the blade root A (Fig. 3.4), which is at a distance  $a\hat{e}_{y1} + e_1\hat{e}_{x1}$  from the hub center. 1-system and 2k-system are parallel.

### 3.5 Preconed, rotating system – 3K

The system 3K, shown in Figs. 3.5.a and 3.5.b, also rotates with blade. This system is obtained by rotating 2K-system by an angle  $\beta_p$  (precone angle) about  $y_{2k}$  axis. The transformation matrix between 2K and 3K systems is given as:

$$[T_{32}] = \begin{bmatrix} 1 & 0 & \beta_p \\ 0 & 1 & 0 \\ -\beta_p & 0 & 1 \end{bmatrix} \quad (3.5)$$

### 3.6 Predrooped, preswept, pitched, blade-fixed rotating system – 4K

The 4K system, shown in Figs. 3.5.a and 3.5.b, is blade fixed system with its origin at pitch bearing B (or flex beam-blade-torque tube junction) of the blade. It may be noted that the pitch axis of the blade is along  $\hat{e}_{x4k}$  direction and the blade reference elastic axis is along the  $\hat{e}_{x4k}$  direction. While changing the control pitch input of the blade, the elastic axis will move on the surface of a cone whose vertex angle is described by the angles  $\beta_s$  and  $\beta_d$  as shown in Fig. 3.5.c.

The 4K system is obtained by the following steps:

Step1: Translating the origin of 3K system by a distance ‘e2’ along  $\hat{e}_{x3k}$

Step2: Then rotating the system by an angle  $\beta_s$  (pre-sweep angle) about  $z_{3k}$  axis.

Step3: Then rotating the system by an angle  $\beta_d$  (pre-droop angle) about  $y_{3k}$  axis.

Step4: Then rotating the system by an angle  $\theta_I$  (pitch input) about  $x_{3k}$ .

The transformation matrix is given as:

$$[T_{43}] = \begin{bmatrix} 1 & -(\beta_s \cos \theta_I + \beta_d \sin \theta_I) & (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \\ (\beta_s \cos \theta_I + \beta_d \sin \theta_I) & \cos \theta_I & \sin \theta_I \\ (-\beta_d \cos \theta_I + \beta_s \sin \theta_I) & -\sin \theta_I & \cos \theta_I \end{bmatrix} \quad (3.6)$$

### 3.7 Undeformed element coordinate system – e

The e-system, shown in Fig. 3.6, has its origin at the inboard node of the finite element. The vector  $\hat{e}_{xe}$ , is aligned with the beam element elastic axis; while the vectors  $\hat{e}_{ye}$  and  $\hat{e}_{ze}$  are cross sectional coordinate axes. For the straight portion of the blade, the  $(\hat{e}_{xe}, \hat{e}_{ye}, \hat{e}_{ze})$  system has the same orientation as  $(\hat{e}_{x4k}, \hat{e}_{y4k}, \hat{e}_{z4k})$  system. For the swept-tip element, the e-system is oriented by rotating the 4K system about  $\hat{e}_{y4k}$  by anhedral angle  $\Lambda_a$ , and then about  $\hat{e}_{z4k}$  by the sweep angle  $\Lambda_s$ .

The transformation matrix between 4k and e-system is given as:

$$\begin{Bmatrix} \hat{e}_{xe} \\ \hat{e}_{ye} \\ \hat{e}_{ze} \end{Bmatrix} = [T_{e4}] \begin{Bmatrix} \hat{e}_{x4k} \\ \hat{e}_{y4k} \\ \hat{e}_{z4k} \end{Bmatrix} \quad (3.7)$$

For the element in the straight portion of the blade

$$[T_{e4}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.8)$$

For the swept-tip element

$$[T_{e4}] = \begin{bmatrix} \cos \Lambda_s \cos \Lambda_a & -\sin \Lambda_s & \cos \Lambda_s \sin \Lambda_a \\ \sin \Lambda_s \cos \Lambda_a & \cos \Lambda_s & \sin \Lambda_s \sin \Lambda_a \\ -\sin \Lambda_a & 0 & \cos \Lambda_a \end{bmatrix} \quad (3.9)$$

Where,  $\Lambda_s$  is the blade tip sweep angle, positive for backward sweep, and  $\Lambda_a$  is the blade tip anhedral angle, positive upward.

### 3.8 Rotating, blade-fixed system – 5K

The 5k system, shown in Fig.3.7, is cross-sectional coordinate system of the  $k^{th}$  blade. In the undeformed state of the blade, both e and 5k systems are parallel. But, the origin of the 5k system is at a distance  $x_k$  from the origin of the e - system. During elastic deformation of the blade, the 5k system translates and rotates with the cross-section. After deformation, the origin of the 5k system, from the origin of 4k system, is at the location given by

$$\left( \sum_{i=1}^{n-1} (l_e)_i \right) \hat{e}_{x4k} + (x_k + u_k) \hat{e}_{xe} + v_k \hat{e}_{ye} + w_k \hat{e}_{ze} \quad (3.10)$$

The transformation matrix between e and 5k system is obtained following a flap – lag sequence of rotation. The Euler angles are respectively  $-\beta_k$  and  $\xi_k$  corresponding to the local slope of the deformed blade in flap and lag directions. The transformation matrix is given by:

$$[T_{5e}] = \begin{bmatrix} \cos \xi_k & \sin \xi_k & 0 \\ -\sin \xi_k & \cos \xi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta_k & 0 & \sin \beta_k \\ 0 & 1 & 0 \\ -\sin \beta_k & 0 & \cos \beta_k \end{bmatrix} \quad (3.11)$$

Since the angle  $\beta_k$  and  $\xi_k$  are of order  $O(\varepsilon)$ , the transformation matrix can be simplified by noting that

$$\sin \theta \approx \theta ; \cos \theta \approx 1$$

The Euler angles can be expressed in the terms of the local slope of elastic deformation of the blade as

$$\begin{aligned} \beta_k &= w'_k \approx \frac{dw_k}{dx} \\ \xi_k &= v'_k \approx \frac{dv_k}{dx} \end{aligned}$$

$$\text{Note: } x = \sum_{i=1}^{n-1} (l_e)_i + x_k$$

Substituting the above relations in the transformation matrix  $[T_{5e}]$  yields

$$[T_{5e}] = \begin{bmatrix} 1 & v'_k & w'_k \\ -v'_k & 1 & -v'_k w'_k \\ -w'_k & 0 & 1 \end{bmatrix} \quad (3.12)$$

### 3.9 Coordinate system-6K

The 6K system, shown in Fig. 3.8, represents the cross-sectional coordinate system in the deformed configuration of the blade. The term  $(\hat{e}_\eta - \hat{e}_\zeta)$  represents the directions of the cross-sectional principal axes. 6K system is obtained by rotating 5K system about  $\hat{e}_{x5k}$  through the angle  $(\phi_k + \theta_k)$ , where  $\theta_k$  represents the geometric twist angle of the cross-section and  $\phi_k$  represents the elastic twist. The transformation relation is given as

$$\begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{Bmatrix}_{5k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_k + \theta_G) & -\sin(\phi_k + \theta_G) \\ 0 & \sin(\phi_k + \theta_G) & \cos(\phi_k + \theta_G) \end{bmatrix} \begin{Bmatrix} \hat{e}_x \\ \hat{e}_\eta \\ \hat{e}_\zeta \end{Bmatrix}_{6k} \quad (3.13)$$

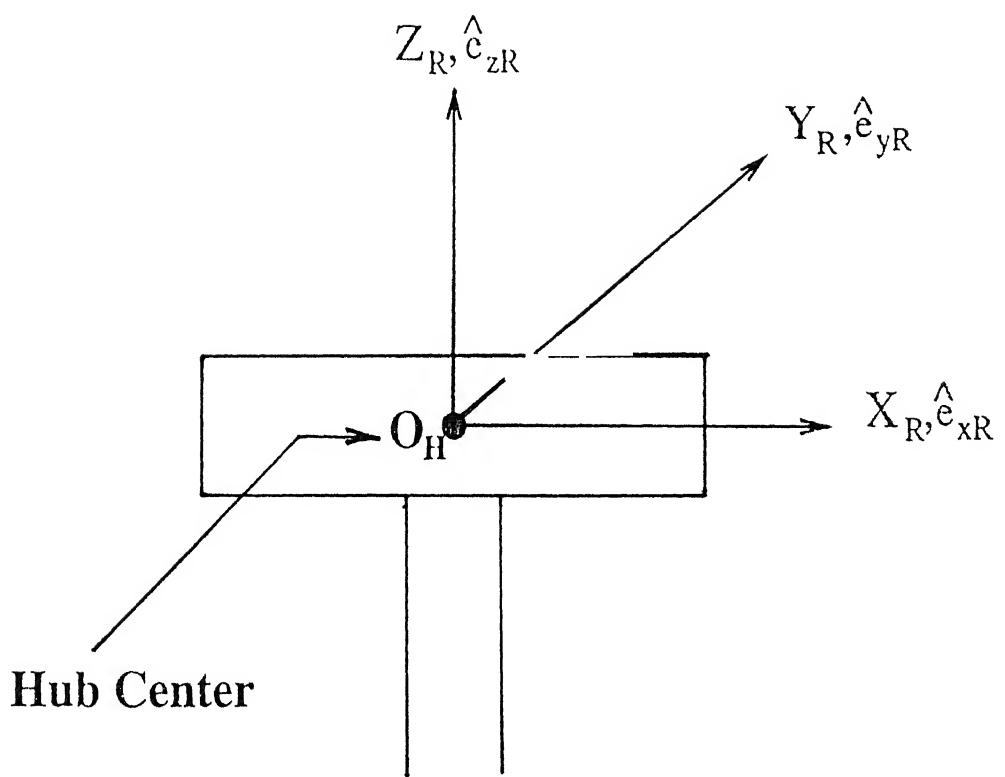


Figure 3.1 Inertial system – R

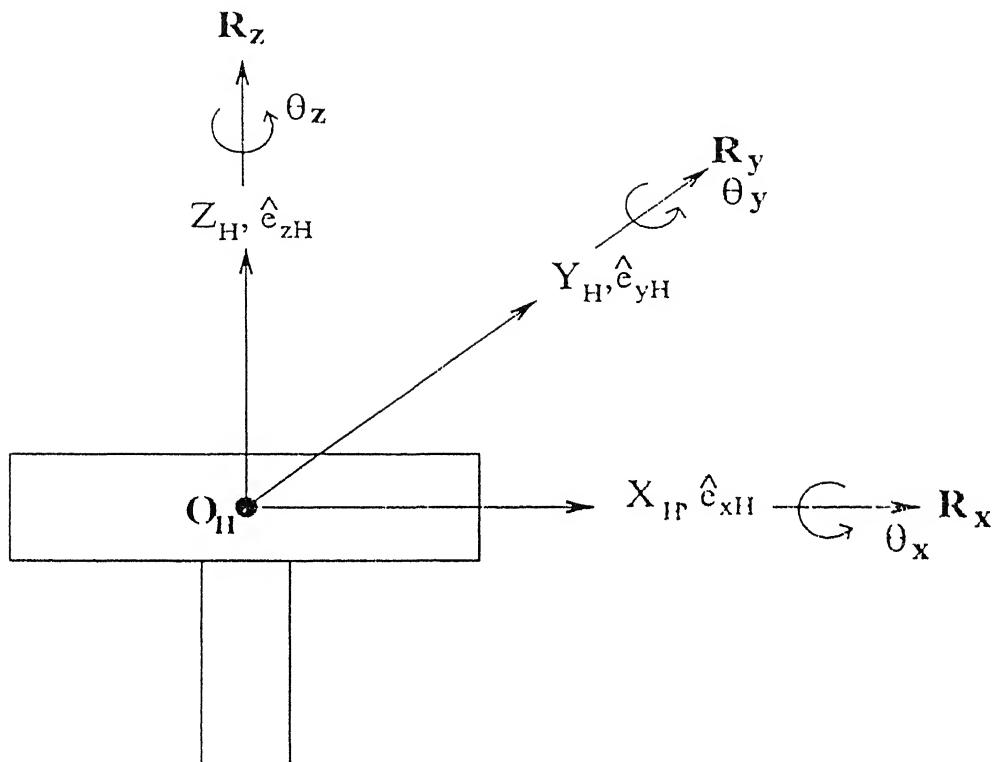


Figure 3.2 Hub fixed co-ordinate system – H

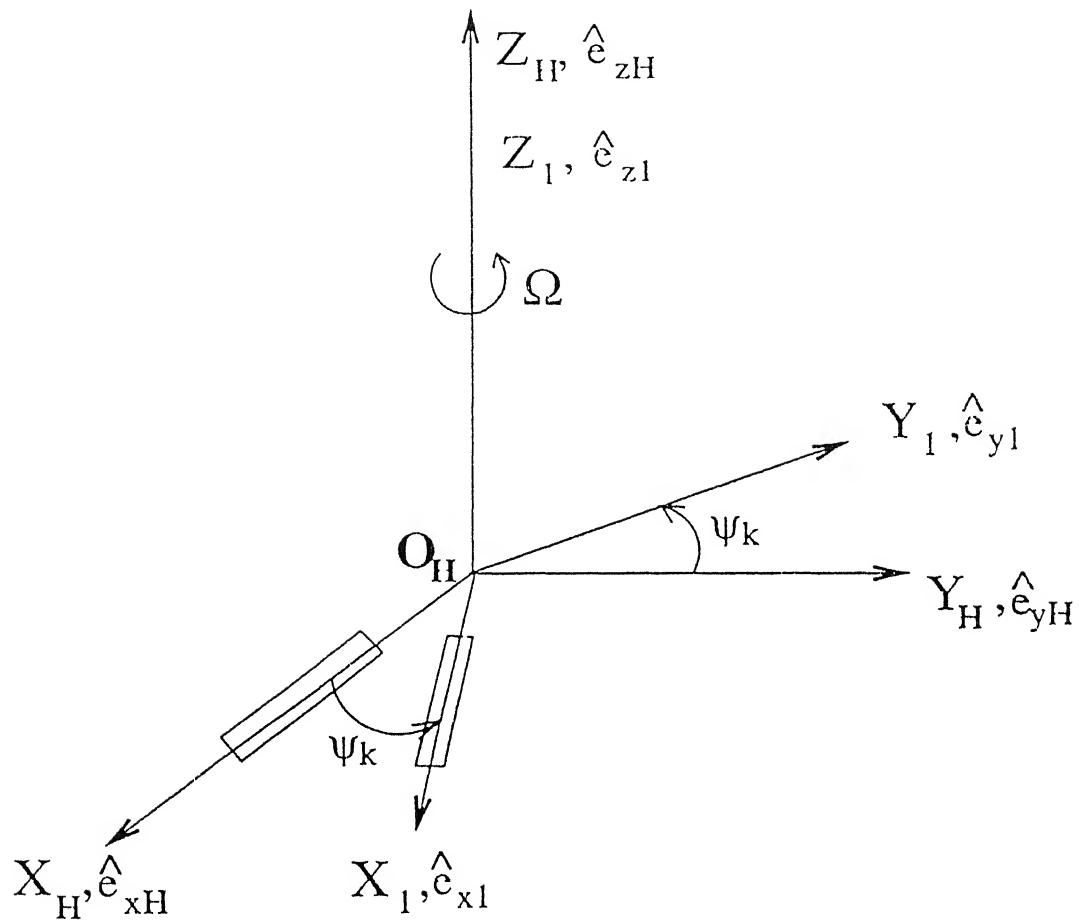


Figure 3.3 Rotating hub system – 1

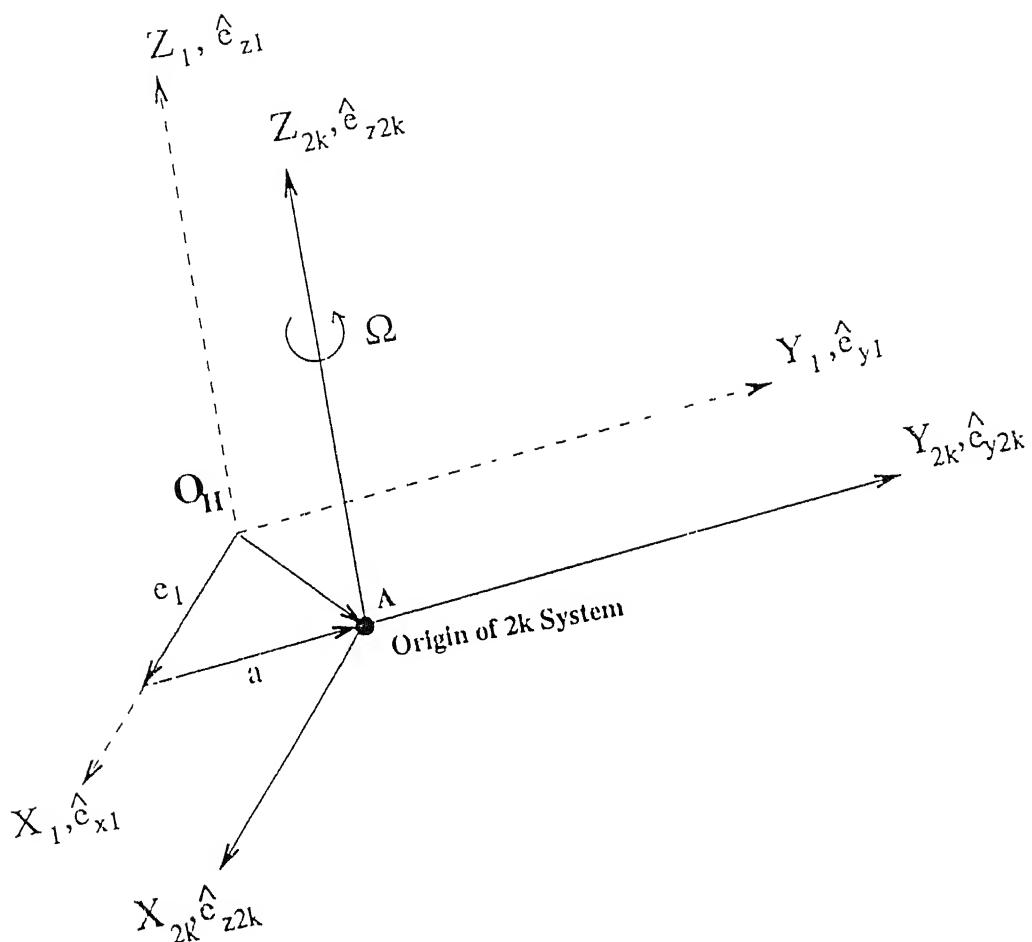
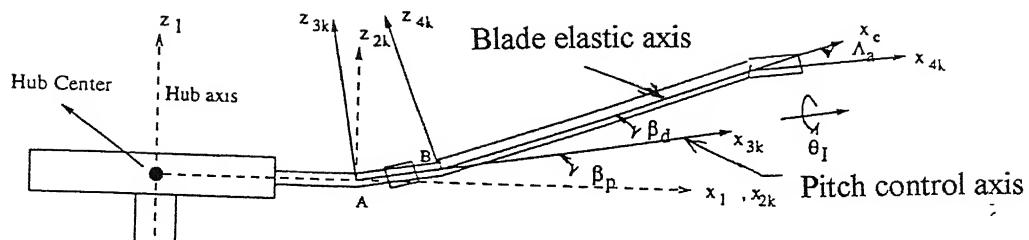
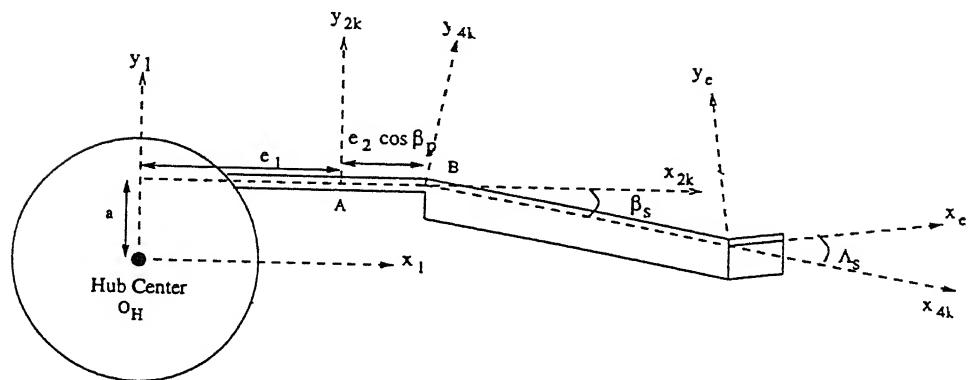


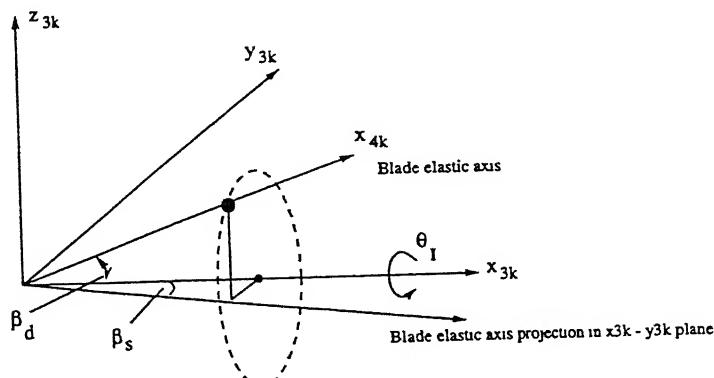
Figure 3.4 Rotating hub system – 2



a) Front view



b) Top view



c) Blade motion due to pitch input

Figure 3.5 Blade co-ordinate system 3k and 4k

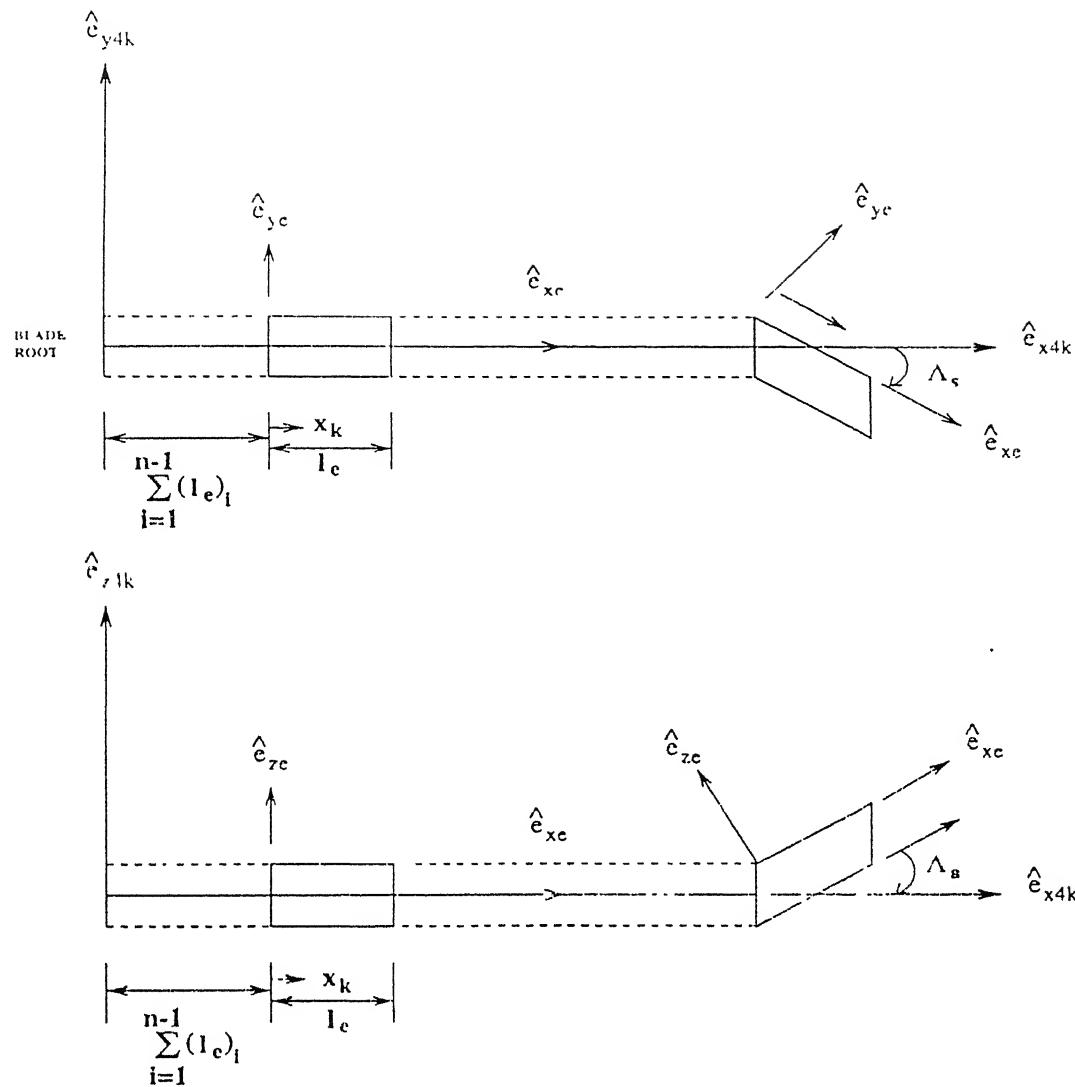
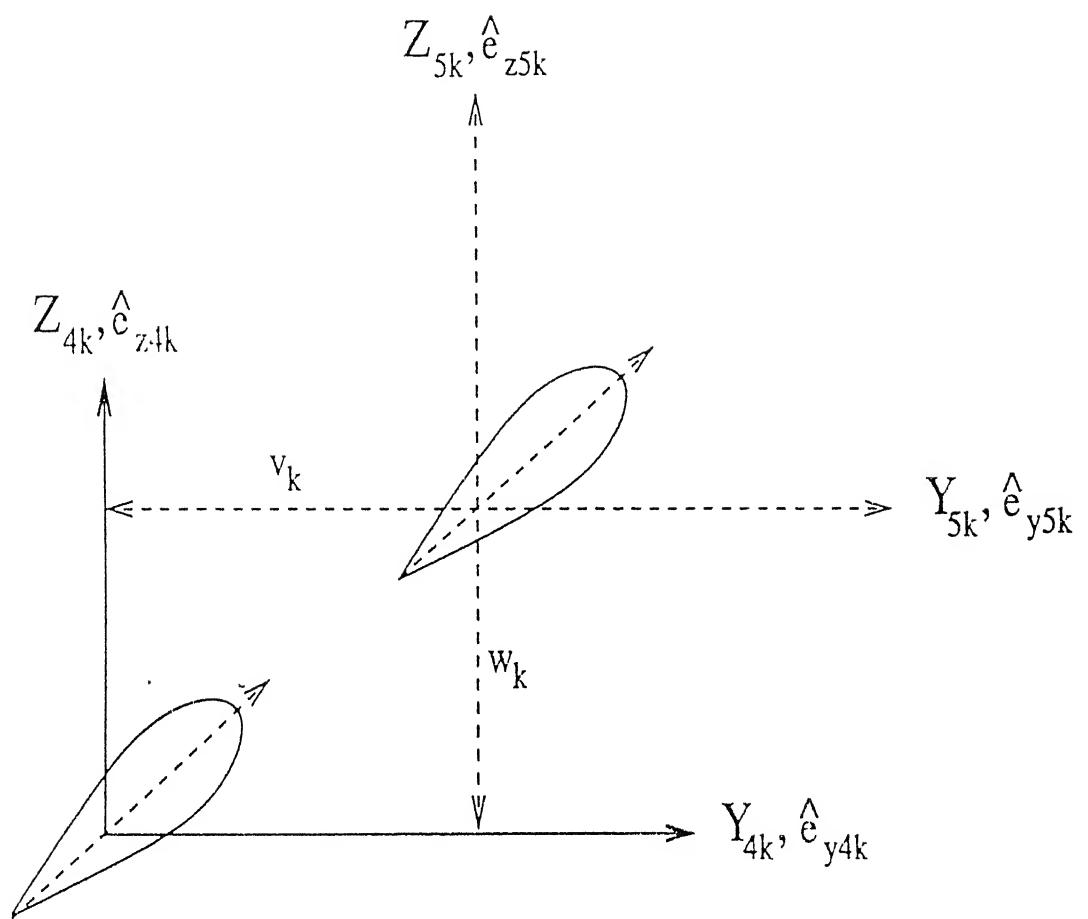
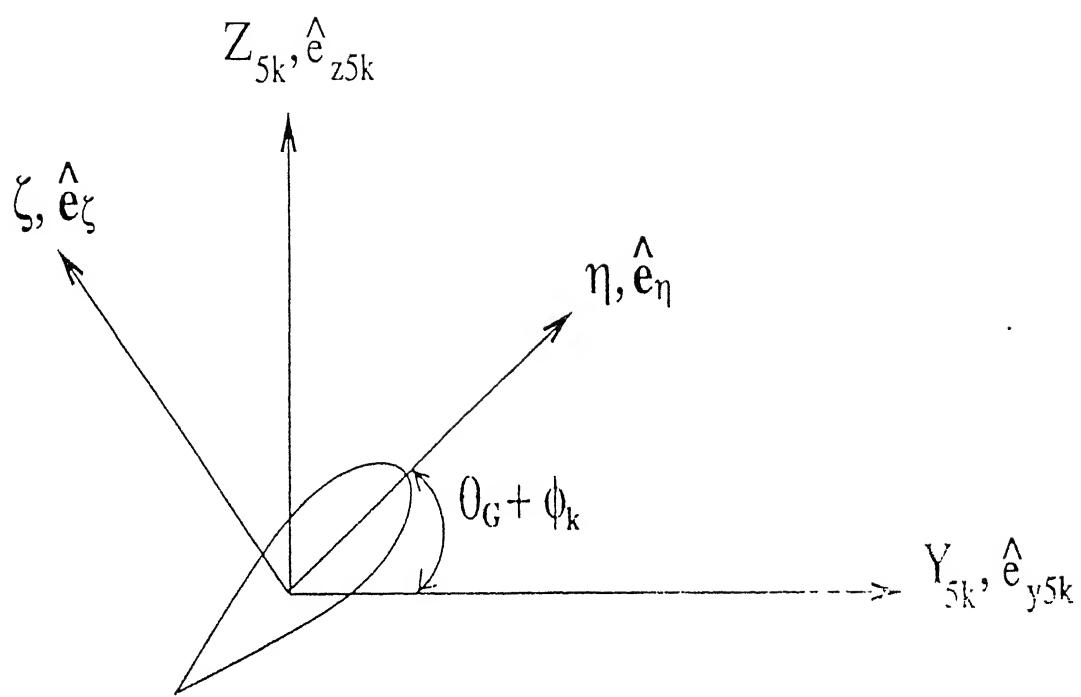


Figure 3.6 Undefomed element co-ordinate system



**Figure 3.7** Rotating blade fixed system - 5k



**Figure 3.8** Cross-sectional principle co-ordinate system

# Chapter 4

## Kinematics

During operation, the rotor blade undergoes in-plane bending (lag), out-of-plane bending (flap), torsion and axial modes. In addition, the hub center has both translational ( $R_x$ ,  $R_y$ ,  $R_z$ ) and rotational ( $\theta_x$ ,  $\theta_y$ ,  $\theta_z$ ) motion. The formulation of inertia operator and aerodynamic operator requires a proper description of kinematics of the blade motion. In this section, an expression for the absolute velocity at any arbitrary point 'p' on the blade is derived.

### 4.1 Position vector of a point

The position vector of any arbitrary point 'p' in the nth finite element of the deformed blade with respect to the hub center  $O_H$ , is given by

$$\vec{r}_p = a\hat{e}_{y1} + e_1\hat{e}_{x2k} + e_2\hat{e}_{x3k} + \sum_{i=1}^{n-1} (l_e)_i \hat{e}_{x4k} + (x_k + u_k)\hat{e}_{xe} + v_k\hat{e}_{ye} + w_k\hat{e}_{ze} + \eta\hat{e}_\eta + \zeta\hat{e}_\zeta \quad (4.1)$$

Transforming all the unit vectors to the 4K-system and neglecting the higher order terms, the position vector can be written as:

$$\begin{aligned} \vec{r}_p = & \hat{e}_{x4k} l \{ -a(\beta_s \cos \theta_I + \beta_d \sin \theta_I) + e_1 + e_2 + \sum_{i=1}^{n-1} (l_e)_i + \cos \Lambda_s \cos \Lambda_a (x_k + u_k) \\ & + v_k \sin \Lambda_s \cos \Lambda_a - w_k \sin \Lambda_a + \eta[-v'_k \cos \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) - w' \cos \Lambda_s \cos \Lambda_a \\ & \sin(\theta_G + \phi_k) + \sin \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) - \sin \Lambda_a \sin(\theta_G + \phi_k)] + \zeta[v'_k \cos \Lambda_s \cos \Lambda_a \\ & \sin(\theta_G + \phi_k) - w'_k \cos \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) - \sin \Lambda_s \cos \Lambda_a \sin(\theta_G + \phi_k) - \sin \Lambda_a \\ & \sin(\theta_G + \phi_k)] \} + \hat{e}_{y4k} l \{ a \cos \theta_I + e_1(\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) + e_2(\beta_s \cos \theta_I + \\ & \beta_d \sin \theta_I) - \sin \Lambda_s (x_k + u_k) + v_k \cos \Lambda_s + \eta[v'_k \sin \Lambda_s \cos(\theta_G + \phi_k) + w'_k \sin \Lambda_s \\ & \sin(\theta_G + \phi_k) + \cos \Lambda_s \cos(\theta_G + \phi_k)] + \zeta[-v'_k \sin \Lambda_s \sin(\theta_G + \phi_k) + w'_k \sin \Lambda_s \cos(\theta_G + \phi_k) \\ & - \cos \Lambda_s \sin(\theta_G + \phi_k)] \} + \hat{e}_{z4k} l \{ -a \sin \theta_I + e_1(-\beta_d \cos \theta_I + \beta_s \sin \theta_I - \beta_p \cos \theta_I) \\ & + e_2(-\beta_d \cos \theta_I + \beta_s \sin \theta_I) + \cos \Lambda_s \sin \Lambda_a (x_k + u_k) + v_k \sin \Lambda_s \sin \Lambda_a + w_k \cos \Lambda_a + \eta \end{aligned}$$

$$[-v'_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) - w'_k \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) + \sin \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k)] + \zeta [v'_k \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) - w'_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) - \sin \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) + \cos \Lambda_a \cos(\theta_G + \phi_k)] \} \quad (4.2)$$

In equation (4.2) all the length quantities are non-dimensionalized with respect to the length of the blade 'l'. For the sake of convenience, the non-dimensional length quantities are referred without an over bar ( ). The expression given in Eq. 4.2 is general, in the sense that for position vector of any point in the straight portion of the blade  $\Lambda_s$  and  $\Lambda_a$  are set equal to zero. Eq. 4.2 can also be written in symbolic form as

$$\vec{r}_p = l[r_x \hat{e}_{x4k} + r_y \hat{e}_{y4k} + r_z \hat{e}_{z4k}] \quad (4.3)$$

## 4.2 Angular velocity vector

The angular velocity vector  $\vec{\omega}$  of  $k^{\text{th}}$  blade consists of three components. They are:

- The variable speed ( $\Omega$ ) of the rotor.
- The rigid body angular velocity ( $\vec{\omega}_{rigid}$ ) of the hub due to perturbational rotation in roll-pitch-yaw ( $\theta_x, \theta_y, \theta_z$ ).
- The angular velocity contribution due to the rate of change of control pitch input  $\Omega_0 \dot{\theta}_I$  to the blade.

$$\vec{\omega} = \Omega \hat{e}_{zH} + \vec{\omega}_{rigid} + \Omega_0 \dot{\theta}_I \hat{e}_{x3k} \quad (4.4)$$

Where,

$$\vec{\omega}_{rigid} = \Omega_0 \dot{\theta}_x \hat{e}_H + \Omega_0 \dot{\theta}_y \hat{e}_H + \Omega_0 \dot{\theta}_z \hat{e}_H \quad (4.5)$$

Where, ( ) indicates differentiation with respect to the non-dimensional time  $\psi_0$ ,

$$\psi_0 = \Omega_0 t$$

Note that

$$\frac{d}{dt}(\ ) = \Omega_0 \frac{d}{d\Omega_0 t}(\ ) = \Omega_0 \frac{d}{d\psi_0}(\ ) = \Omega_0 (\dot{\ })$$

Transforming the all the unit vectors of Eq. 4.4 to the 4K-system and neglecting the higher order terms, the angular velocity of  $k^{\text{th}}$  blade can be written in symbolic form as:

$$\vec{\omega} = \Omega_0 [\omega_x \hat{e}_{x4k} + \omega_y \hat{e}_{y4k} + \omega_z \hat{e}_{z4k}] \quad (4.6)$$

Where,

$$\omega_x = \left[ \left( \frac{\Omega}{\Omega_0} \right) (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \dot{\theta}_x \cos \psi_k + \dot{\theta}_y \sin \psi_k + \dot{\theta}_z (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \dot{\theta}_I \right] \quad (4.7)$$

$$\omega_y = \left[ \left( \frac{\Omega}{\Omega_0} \right) \sin \theta_I - \dot{\theta}_x \sin \psi_k \cos \theta_I + \dot{\theta}_y \cos \psi_k \cos \theta_I + \dot{\theta}_z \sin \theta_I + \dot{\theta}_I (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right] \quad (4.8)$$

$$\omega_z = \left[ \left( \frac{\Omega}{\Omega_0} \right) \cos \theta_I + \dot{\theta}_x \sin \theta_I \sin \psi_k - \dot{\theta}_y \cos \psi_k \sin \theta_I + \dot{\theta}_z \cos \theta_I + \dot{\theta}_I (-\beta_d \cos \theta_I + \beta_s \sin \theta_I) \right] \quad (4.9)$$

### 4.3 Velocity at a point 'p'

The absolute velocity vector  $\vec{V}$ , at point 'p' on the deformed beam can be written as:

$$\vec{V} = \vec{V}_H + \dot{\vec{r}}_p + (\vec{\omega} \times \vec{r}_p) \quad (4.10)$$

Where,  $\vec{V}_H$  is the rigid body perturbational translation of the hub center  $O_H$ , which is given as:

$$\vec{V}_H = \Omega_0 l [\dot{R}_x \hat{e}_{xR} + \dot{R}_y \hat{e}_{yR} + \dot{R}_z \hat{e}_{zR}] \quad (4.11)$$

Transforming all the unit vectors to 4K-system

$$\vec{V}_H = [(V_H)_x \hat{e}_{x4k} + (V_H)_y \hat{e}_{y4k} + (V_H)_z \hat{e}_{z4k}] \quad (4.12)$$

Where,

$$(V_H)_x = \Omega_0 l \dot{R}_x \{ \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \} + \Omega_0 l \dot{R}_y \{ \sin \psi_k (-\cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I)) \} + \Omega_0 l \dot{R}_z (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) \quad (4.13)$$

$$(V_H)_y = \Omega_0 l \dot{R}_x \{ \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) \} - \sin \psi_k \cos \theta_I + \Omega_0 l \dot{R}_y \{ \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) \} + \cos \theta_I \cos \psi_k + \Omega_0 l \dot{R}_z (\sin \theta_I) \quad (4.14)$$

$$(V_H)_z = \Omega_0 l \dot{R}_x \{ \cos \psi_k (-\beta_d \cos \theta_I + \beta_s \sin \theta_I - \beta_p \cos \theta_I) \} + \sin \psi_k \sin \theta_I + \Omega_0 l \dot{R}_y \{ -\cos \psi_k \sin \theta_I + \sin \psi_k (-\beta_d \cos \theta_I + \beta_s \sin \theta_I - \beta_p \cos \theta_I) \} + \Omega_0 l \dot{R}_z \{ \cos \theta_I \} \quad (4.15)$$

and:

$$\begin{aligned} \dot{\vec{r}}_p &= \hat{e}_{x4k} l \Omega_0 \left[ -\alpha \dot{\theta}_I (-\beta_s \sin \theta_I + \beta_d \cos \theta_I) + \cos \Lambda_s \cos \Lambda_a \dot{\theta}_k + \sin \Lambda_s \cos \Lambda_a \dot{\phi}_k \right. \\ &\quad - \dot{\omega}_k \sin \Lambda_a + n \left\{ -\dot{\psi}'_k \cos \Lambda_s \cos \Lambda_a \cos(\theta_G + \theta_k) + \dot{\psi}'_k \dot{\theta}_k \cos \Lambda_s \cos \Lambda_a \sin(\theta_G + \phi_k) \right\} \\ &\quad \left. - \dot{\omega}'_k \cos \Lambda_s \cos \Lambda_a \sin(\theta_G + \phi_k) - \omega'_k \cos \Lambda_s \cos \Lambda_a \dot{\phi}_k \cos(\theta_G + \phi_k) - \dot{\phi}_k \sin \Lambda_s \cos \Lambda_a \right] \end{aligned}$$

$$\begin{aligned}
& \sin(\theta_G + \phi_k) - \dot{\phi}_k \sin \Lambda_a \cos(\theta_G + \phi_k) \} + \zeta \{ \dot{v}'_k \cos \Lambda_s \cos \Lambda_a \sin(\theta_G + \phi_k) \\
& + v'_k \dot{\phi}_k \cos \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) - \dot{\omega}'_k \cos \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) + \omega'_k \dot{\phi}_k \\
& \cos \Lambda_s \cos \Lambda_a \sin(\theta_G + \phi_k) - \dot{\phi}_k \sin \Lambda_s \cos \Lambda_a \cos(\theta_G + \phi_k) + \dot{\phi}_k \sin \Lambda_a \sin(\theta_G + \phi_k) \} \} \\
& + \hat{e}_{y4k} l \Omega_0 \left[ -a \dot{\theta}_I \sin \theta_I + e_1 \dot{\theta}_I (-\beta_s \sin \theta_I + \beta_d \cos \theta_I - \beta_p \cos \theta_I) + e_2 \dot{\theta}_I (-\beta_s \sin \theta_I + \beta_d \cos \theta_I) \right. \\
& - \dot{u}_k \sin \Lambda_s + \dot{v}_k \cos \Lambda_s + n \{ \dot{v}'_k \sin \Lambda_s \cos(\theta_G + \phi_k) - v'_k \dot{\phi}_k \sin \Lambda_s \sin(\theta_G + \phi_k) + \dot{\omega}'_k \sin \Lambda_s \sin(\theta_G + \phi_k) \\
& + \omega'_k \dot{\phi}_k \sin \Lambda_s \cos(\theta_G + \phi_k) - \dot{\phi}_k \cos \Lambda_s \sin(\theta_G + \phi_k) \} + \xi \{ -v'_k \sin \Lambda_s \sin(\theta_G + \phi_k) - v'_k \dot{\phi}_k \sin \Lambda_s \cos(\theta_G + \phi_k) \\
& + \omega'_k \sin \Lambda_s \cos(\theta_G + \phi_k) - \omega'_k \dot{\phi}_k \sin \Lambda_s \sin(\theta_G + \phi_k) - \dot{\phi}_k \cos \Lambda_s \cos(\theta_G + \phi_k) \} \} + \hat{e}_{z4k} l \Omega_0 \left[ -a \dot{\theta}_I \cos \theta_I + e_1 \dot{\theta}_I (\beta_d \sin \theta_I + \beta_s \cos \theta_I + \beta_p \sin \theta_I) + e_2 \dot{\theta}_I (\beta_d \sin \theta_I + \beta_s \cos \theta_I) + \dot{u}_k \cos \Lambda_s \sin \Lambda_a + \dot{v}_k \sin \Lambda_s \sin \Lambda_a \right. \\
& + \dot{\omega}_k \cos \Lambda_a + n \{ -\dot{v}'_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) + v'_k \dot{\phi}'_k \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) - \dot{\omega}'_k \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) \\
& - \omega'_k \dot{\phi}_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) - \dot{\phi}_k \sin \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) + \dot{\phi}_k \cos \Lambda_a \cos(\theta_G + \phi_k) \} + \xi \{ \dot{v}'_k \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) + v'_k \dot{\phi}_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) - \dot{\omega}'_k \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) + \omega'_k \dot{\phi}_k \\
& \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) - \dot{\phi}_k \sin \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) - \dot{\phi}_k \cos \Lambda_a \sin(\theta_G + \phi_k) \} \} \]
\end{aligned} \tag{4.16}$$

and

$$\vec{\omega} \vec{x} \vec{r} = \begin{bmatrix} \hat{e}_{x4k} & \hat{e}_{y4k} & \hat{e}_{z4k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{bmatrix} \tag{4.17}$$

$$(\vec{\omega} \vec{x} \vec{r}) = \hat{e}_{x4k} (\omega_y r_z - r_y \omega_z) - \hat{e}_{y4k} (\omega_x r_z - r_x \omega_z) + \hat{e}_{z4k} (\omega_x r_y - r_x \omega_y) \tag{4.18}$$

Where,  $r_x$ ,  $r_y$ ,  $r_z$  and  $\omega_x$ ,  $\omega_y$ ,  $\omega_z$  are the x, y, and z components of  $\vec{r}_p$  and  $\vec{\omega}$  respectively.

Substituting various quantities in Eq. 4.10 from Eqs. 4.2, 4.7-4.9 and 4.13-4.18, the velocity at point 'P' is obtained. This can be written in symbolic form as:

$$\vec{V} = \Omega_0 [V_x \hat{e}_{x4k} + V_y \hat{e}_{y4k} + V_z \hat{e}_{z4k}] \tag{4.19}$$

The components of the velocity vector are given below

$$V_u = (V_H)_u + (\dot{r}_p)_u + (\omega_y r_z - \omega_z r_y)$$

$$V_v = (V_H)_v + (\dot{r}_p)_v - (\omega_x r_z - \omega_z r_x)$$

$$V_z = (V_H)_z + (\dot{r}_p)_z + (\omega_x r_z - \omega_y r_x) \quad (4.20)$$

$$\begin{aligned}
(V_x) = & \Omega_0 l [\dot{R}_x (\cos \psi_k + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I)) + \dot{R}_y (-\cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) + \sin \psi_k) \\
& + \dot{R}_z (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \alpha \dot{\theta}_I (\beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \dot{u}_k + \sin \Lambda_s \cos \Lambda_a \dot{v}_k \\
& - \dot{\omega}_k \sin \Lambda_a - [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (\dot{v}'_k \cos \Lambda_s \cos \Lambda_a + \omega'_k \dot{\phi}_k \cos \Lambda_s \cos \Lambda_a + \phi_k \sin \Lambda_a) \\
& + [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (\dot{v}'_k \dot{\phi}_k \cos \Lambda_s \cos \Lambda_a - \dot{\omega}'_k \cos \Lambda_s \cos \Lambda_a - \dot{\phi}_k \sin \Lambda_s \cos \Lambda_a) - \alpha \left( \frac{\Omega}{\Omega_0} \right) \\
& - \left( \frac{\Omega}{\Omega_0} \right) [\beta_p \sin 2\theta_I + \beta_s \cos 2\theta_I] (e_1 + e_2) + \left( \frac{\Omega}{\Omega_0} \right) (x_k + u_k) [\cos \theta_I \sin \Lambda_s + \sin \theta_I \cos \Lambda_s \sin \Lambda_a] + \left( \frac{\Omega}{\Omega_0} \right) \\
& v_k [\sin \theta_I \sin \Lambda_s \sin \Lambda_a - \cos \theta_I \cos \Lambda_s] + \left( \frac{\Omega}{\Omega_0} \right) \omega_k \sin \theta_I \cos \Lambda_a + [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (-v'_k \\
& \left( \frac{\Omega}{\Omega_0} \right) [\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \sin \Lambda_s \cos \theta_I] + \left( \frac{\Omega}{\Omega_0} \right) [\sin \theta_I \sin \Lambda_s \sin \Lambda_a - \cos \theta_I \cos \Lambda_s]) + [\eta \sin(\theta_G + \phi_k) \\
& + \xi \cos(\theta_G + \phi_k)] \left[ -\omega'_k \left( \frac{\Omega}{\Omega_0} \right) [\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \cos \theta_I \sin \Lambda_s] + \left( \frac{\Omega}{\Omega_0} \right) \sin \theta_I \cos \Lambda_a \right] - \dot{\theta}_x e_1 \sin \psi_k \\
& [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I - \beta_p] - \dot{\theta}_x e_2 \sin \psi_k [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I] - \dot{\theta}_x \sin \psi_k (x_k + u_k) [\cos \theta_I \cos \Lambda_s \\
& \sin \Lambda_a - \sin \theta_I \sin \Lambda_s] - \dot{\theta}_x v_k \sin \psi_k [\cos \theta_I \sin \Lambda_s \sin \Lambda_a + \sin \theta_I \cos \Lambda_s] - \dot{\theta}_x \omega_k \sin \psi_k \cos \theta_I \cos \Lambda_a + \dot{\theta}_x \\
& [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (v'_k \sin \psi_k (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s) - \sin \psi_k [\cos \theta_I \sin \Lambda_s \sin \Lambda_a \\
& + \sin \theta_I \cos \Lambda_s]) + \dot{\theta}_x [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (\omega'_k \sin \psi_k [\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s] - \\
& \sin \psi_k \cos \theta_I \cos \Lambda_a) + \dot{\theta}_y e_1 \cos \psi_k [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I - \beta_p] + \dot{\theta}_y e_2 \sin \psi_k [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I] \\
& + \dot{\theta}_y \cos \psi_k (x_k + u_k) [\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s] + \dot{\theta}_y \cos \psi_k v_k [\cos \theta_I \sin \Lambda_s \sin \Lambda_a + \sin \theta_I \cos \Lambda_s] \\
& + \dot{\theta}_y \cos \psi_k \omega_k (\cos \theta_I \cos \Lambda_a) + \dot{\theta}_y [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (-\cos \psi_k v'_k (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \\
& \sin \Lambda_s) + \cos \psi_k (\cos \theta_I \sin \Lambda_s \sin \Lambda_a + \sin \theta_I \cos \Lambda_s)) + \dot{\theta}_y [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (-\omega'_k \cos \psi_k \\
& (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s) + \cos \psi_k \cos \theta_I \cos \Lambda_a) - \alpha \dot{\theta}_z - (e_1 + e_2) \dot{\theta}_z (\beta_d \sin 2\theta_I + \beta_s \cos 2\theta_I) \\
& + \dot{\theta}_z (x_k + u_k) [\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \cos \theta_I \sin \Lambda_s] + \dot{\theta}_z v_k (\sin \theta_I \sin \Lambda_s \sin \Lambda_a - \cos \theta_I \cos \Lambda_s) + \dot{\theta}_z \omega_k \cos \Lambda_a \\
& \sin \theta_I + \dot{\theta}_z [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (-v'_k (\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \sin \Lambda_s \cos \theta_I) + \sin \theta_I \sin \Lambda_s \sin \Lambda_a)
\end{aligned}$$

$$\begin{aligned}
& -\cos\theta_I \cos\Lambda_s) + \dot{\theta}_z [n \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (-\omega'_k (\sin\theta_I \cos\Lambda_s \sin\Lambda_a + \sin\Lambda_s \cos\theta_I) + \sin\theta_I \\
& \cos\Lambda_a) - a \dot{\theta}_I [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I] + \dot{\theta}_I (x_k + u_k)] [\beta_s (\cos\theta_I \cos\Lambda_s \sin\Lambda_a + \sin\theta_I \sin\Lambda_s) + \beta_d (\sin\theta_I \\
& \cos\Lambda_s \sin\Lambda_a - \cos\theta_I \sin\Lambda_s)] + \dot{\theta}_I v_k [\beta_s (\cos\theta_I \sin\Lambda_s \sin\Lambda_a - \sin\theta_I \cos\Lambda_s) + \beta_d (\sin\theta_I \sin\Lambda_s \sin\Lambda_a + \\
& \cos\theta_I \cos\Lambda_s)] + \dot{\theta}_I \omega_k \cos\Lambda_a [\beta_s \cos\theta_I + \beta_d \sin\theta_I] + \theta_I [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (-v'_k [\beta_s (\cos\theta_I \\
& \cos\Lambda_s \sin\Lambda_a + \sin\theta_I \sin\Lambda_s) + \beta_d (\sin\theta_I \cos\Lambda_s \sin\Lambda_a - \cos\theta_I \sin\Lambda_s)] + [\beta_s (\cos\theta_I \sin\Lambda_s \sin\Lambda_a - \sin\theta_I \\
& \cos\Lambda_s + \beta_d (\sin\theta_I \sin\Lambda_s \sin\Lambda_a + \cos\theta_I \cos\Lambda_s)])] + \dot{\theta}_I [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (-\omega'_k [\beta_s (\cos\theta_I \\
& \cos\Lambda_s \sin\Lambda_a + \sin\theta_I \sin\Lambda_s) + \beta_d (\sin\theta_I \cos\Lambda_s \sin\Lambda_a - \cos\theta_I \sin\Lambda_s)] + \cos\Lambda_a (\beta_s \cos\theta_I + \beta_d \sin\theta_I)) \]
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
(V_y) = & \Omega_0 I [\dot{R}_x (\cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) - \sin \psi_k \cos \theta_I) + \dot{R}_y (\cos \psi_k \cos \theta_I + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I)) + \dot{R}_z \sin \theta_I - \alpha \dot{\theta}_I \sin \theta_I + e_1 \dot{\theta}_I (-\beta_s \sin \theta_I + \beta_d \cos \theta_I - \beta_p \cos \theta_I) + e_2 \dot{\theta}_I (-\beta_s \sin \theta_I + \beta_d \cos \theta_I) - \dot{u}_k \sin \Lambda_s + \dot{v}_k \cos \Lambda_s + [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] (\dot{v}'_k \sin \Lambda_s + \sin \Lambda_s \omega'_k \dot{\phi}_k) + [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (-\dot{v}'_k \phi'_k \sin \Lambda_s + \dot{\omega}'_k \sin \Lambda_s - \dot{\phi}'_k \cos \Lambda_s) + \alpha \left(\frac{\Omega}{\Omega_0}\right) [\beta_p \sin \theta_I - \beta_s] + \left(\frac{\Omega}{\Omega_0}\right) (e_1 + e_2 + \sum_{i=1}^{n-1} u_i) \cos \theta_I - \left(\frac{\Omega}{\Omega_0}\right) (x_k + u_k) \cos \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a] - \left(\frac{\Omega}{\Omega_0}\right) v_k \sin \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a] - \left(\frac{\Omega}{\Omega_0}\right) \omega_k [\cos \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \sin \Lambda_a \cos \theta_I] - \left(\frac{\Omega}{\Omega_0}\right) [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{-\dot{v}'_k \cos \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a]\} - \left(\frac{\Omega}{\Omega_0}\right) [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{-\omega'_k \cos \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a] + [\cos \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \theta_I \sin \Lambda_a]\} - \dot{\theta}_x a \sin \theta_I (-\cos \psi_k + \beta_s \cos \theta_I \sin \psi_k + \beta_d \sin \theta_I \sin \psi_k) - \dot{\theta}_x e_1 [\cos \psi_k (-\beta_d \cos \theta_I + \beta_s \sin \theta_I) - \sin \psi_k \sin \theta_I] - e_2 \dot{\theta}_x [\cos \psi_k (-\beta_d \cos \theta_I + \beta_s \sin \theta_I) - \sin \psi_k \sin \theta_I] + \dot{\theta}_x \sum_{i=1}^{n-1} u_i \sin \psi_k \sin \theta_I - \dot{\theta}_x (x_k + u_k) [\cos \Lambda_s \cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_s \cos \Lambda_a] - \dot{\theta}_x v_k [\sin \Lambda_s \cos \psi_k \sin \Lambda_a - \sin \Lambda_s \sin \theta_I \sin \psi_k \cos \Lambda_a] - \dot{\theta}_x \omega_k [\cos \psi_k \cos \Lambda_a + \sin \theta_I \sin \psi_k \sin \Lambda_a] - \dot{\theta}_x [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{-\dot{v}'_k \cos \Lambda_s (\cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_a) + \sin \Lambda_s (\cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_a)\} - \dot{\theta}_x [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{-\omega'_k \cos \Lambda_s [\cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_a] + (\cos \psi_k \cos \Lambda_a - \sin \theta_I \sin \psi_k \sin \Lambda_a)\} + \dot{\theta}_y a \sin \theta_I (\sin \psi_k + \beta_s \cos \theta_I \cos \psi_k + \beta_d \cos \psi_k \sin \theta_I) + \dot{\theta}_y e_1 (\beta_d \cos \theta_I \sin \psi_k - \beta_s \sin \theta_I \sin \psi_k + \beta_p \cos \theta_I \sin \psi_k - \cos \psi_k \sin \theta_I) + \dot{\theta}_y e_2 (\beta_d \cos \theta_I \sin \psi_k - \beta_s \sin \theta_I \sin \psi_k - \cos \psi_k \sin \theta_I) - \dot{\theta}_y \sum_{i=1}^{n-1} u_i \cos \psi_k \sin \theta_I - \dot{\theta}_y (x_k + u_k) \cos \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a) - \dot{\theta}_y v_k (\sin \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a) - \dot{\theta}_y \omega_k (\sin \psi_k \cos \Lambda_a - \sin \theta_I \cos \psi_k \sin \Lambda_a) - \dot{\theta}_y [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{-\dot{v}'_k \cos \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a) + \sin \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a)\} - \dot{\theta}_y [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{-\omega'_k \cos \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a) + (\sin \psi_k \cos \Lambda_a - \sin \theta_I \cos \psi_k \sin \Lambda_a)\} - \dot{\theta}_z a (\beta_s - \beta_p \sin \theta_I) + \dot{\theta}_z \left(e_1 + e_2 + \sum_{i=1}^{n-1} u_i\right) \cos \theta_I - \dot{\theta}_z (x_k + u_k) \cos \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a] - \dot{\theta}_z \omega_k [\cos \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \sin \Lambda_a \cos \theta_I] - \dot{\theta}_z [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{-\dot{v}'_k \cos \Lambda_s (\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a) + \sin \Lambda_s (\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a)\} - \dot{\theta}_z [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{-\omega'_k \cos \Lambda_s [\sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a] + (\cos \theta_I \sin \Lambda_a + \cos \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I))\} + \dot{\theta}_z a \sin \theta_I + \dot{\theta}_z e_1 \beta_p \cos \theta_I + \sum_{i=1}^{n-1} u_i \dot{\theta}_z (-\beta_d \cos \theta_I + \beta_s \sin \theta_I) - \dot{\theta}_z (x_k + u_k) \cos \Lambda_s [\cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I) + \sin \Lambda_a] - v_k \dot{\theta}_z
\end{aligned}$$

$$\begin{aligned}
& \sin \Lambda_s [\sin \Lambda_a + \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I)] - \omega_k \dot{\theta}_I [\cos \Lambda_a + \sin \Lambda_a (\beta_s \sin \theta_I - \beta_d \cos \theta_I)] - \dot{\theta}_I \\
& [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{ -v'_k \cos \Lambda_s [\sin \Lambda_a + \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I)] + \sin \Lambda_s [\sin \Lambda_a + \\
& \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I)] \} - \theta_I [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{ -\omega'_k \cos \Lambda_s (\sin \Lambda_a - \cos \Lambda_a \\
& (\beta_s \sin \theta_I - \beta_d \cos \theta_I)) + (\cos \Lambda_a + \sin \Lambda_a (\beta_s \sin \theta_I - \beta_d \cos \theta_I)) \}
\end{aligned}
\tag{4.22}$$

$$\begin{aligned}
(V_z) = & \Omega_0 l [\dot{R}_x [\cos \psi_k (-\beta_d \cos \theta_I + \beta_s \sin \theta_I - \beta_p \cos \theta_I) + \sin \psi_k \sin \theta_I] + \dot{R}_y [-\cos \psi_k \sin \theta_I + \sin \psi_k (-\beta_d \cos \theta_I \\
& + \beta_s \sin \theta_I - \beta_p \cos \theta_I)] + \dot{R}_z \cos \theta_I - a \dot{\theta}_I \cos \theta_I + e_1 \dot{\theta}_I (\beta_d \sin \theta_I + \beta_s \cos \theta_I + \beta_p \sin \theta_I) + e_2 \dot{\theta}_I (\beta_d \\
& \sin \theta_I + \beta_s \cos \theta_I) + \dot{u}_k \cos \Lambda_s \sin \Lambda_a + \dot{v}_k \sin \Lambda_s \sin \Lambda_a + \dot{\omega}_k \cos \Lambda_a + [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \\
& (-v'_k \cos \Lambda_s \sin \Lambda_a - \omega'_k \dot{\phi}_k \cos \Lambda_s \sin \Lambda_a + \dot{\phi}_k \cos \Lambda_a) + [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] (v'_k \dot{\phi}_k \cos \Lambda_s \sin \Lambda_a \\
& - \dot{\omega}'_k \cos \Lambda_s \sin \Lambda_a - \dot{\phi}_k \sin \Lambda_s \sin \Lambda_a) + \left(\frac{\Omega}{\Omega_0}\right) a (\beta_d + \beta_p \cos \theta_I) - \left(\frac{\Omega}{\Omega_0}\right) \left(e_1 + e_2 + \sum_{i=1}^{n-1} u_i\right) \sin \theta_I - \left(\frac{\Omega}{\Omega_0}\right) (x_k + u_k) \\
& [\sin \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I] + \left(\frac{\Omega}{\Omega_0}\right) v'_k [\cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \\
& \sin \Lambda_s \cos \Lambda_a \sin \theta_I] + \left(\frac{\Omega}{\Omega_0}\right) \omega_k \sin \Lambda_a \sin \theta_I + \left(\frac{\Omega}{\Omega_0}\right) [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{ v'_k [\sin \Lambda_s (\beta_p + \beta_d \cos \theta_I \\
& - \beta_s \sin \theta_I) + \sin \theta_I \cos \Lambda_s \cos \Lambda_a] + \cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \sin \Lambda_s \cos \Lambda_a \sin \theta_I \} + \left(\frac{\Omega}{\Omega_0}\right) \\
& [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{ \omega'_k [\sin \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I] + \sin \theta_I \\
& \sin \Lambda_a \} + \dot{\theta}_x a \cos \theta_I (\cos \psi_k - \beta_s \sin \psi_k \cos \theta_I - \beta_d \sin \psi_k \sin \theta_I) + \dot{\theta}_x e_1 (\sin \psi_k \cos \theta_I + \beta_s \cos \psi_k \cos \theta_I + \\
& \beta_d \cos \psi_k \sin \theta_I - \beta_p \cos \psi_k \sin \theta_I) + \dot{\theta}_x e_2 (\beta_s \cos \psi_k \cos \theta_I + \beta_d \cos \psi_k \sin \theta_I + \cos \theta_I \sin \psi_k) + \dot{\theta}_x \sum_{i=1}^{n-1} u_i \\
& \sin \psi_k \cos \theta_I + \theta_x (x_k + u_k) [\sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a - \cos \psi_k \sin \Lambda_s] + \dot{\theta}_x v'_k [\cos \psi_k \cos \Lambda_s + \sin \psi_k \cos \theta_I \\
& \sin \Lambda_s \cos \Lambda_a] - \dot{\theta}_x \omega_k \sin \psi_k \cos \theta_I \sin \Lambda_a + \dot{\theta}_x [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{ v'_k (\cos \psi_k \sin \Lambda_s - \sin \psi_k \\
& \sin \psi_k \cos \Lambda_s \cos \Lambda_a) + (\cos \psi_k \cos \Lambda_s + \sin \psi_k \cos \theta_I \sin \Lambda_s \cos \Lambda_a) \} + \dot{\theta}_x [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \\
& \{ \omega'_k (\cos \psi_k \sin \Lambda_s - \sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) - \sin \psi_k \cos \theta_I \sin \Lambda_a \} + \dot{\theta}_y a \cos \theta_I (\sin \psi_k + \beta_s \cos \psi_k \cos \theta_I \\
& \beta_d \sin \theta_I \cos \psi_k) + \dot{\theta}_y e_1 (\beta_s \sin \psi_k \cos \theta_I + \beta_d \sin \psi_k \sin \theta_I - \beta_p \sin \psi_k \sin \theta_I - \cos \psi_k \cos \theta_I) + \dot{\theta}_y e_2 \\
& [\beta_s \sin \psi_k \cos \theta_I + \beta_d \sin \psi_k \sin \theta_I - \cos \psi_k \cos \theta_I] - \dot{\theta}_y \sum_{i=1}^{n-1} u_i \cos \psi_k \cos \theta_I - \dot{\theta}_y (x_k + u_k) [\sin \psi_k \sin \Lambda_s + \\
& \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a] + \dot{\theta}_y v'_k [\sin \psi_k \cos \Lambda_s - \cos \psi_k \cos \theta_I \sin \Lambda_s \cos \Lambda_a] + \dot{\theta}_y \cos \psi_k \omega_k \cos \theta_I \sin \Lambda_a \\
& + \dot{\theta}_y [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{ v'_k (\sin \psi_k \sin \Lambda_s + \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) + (\cos \Lambda_s \sin \psi_k - \\
& \sin \Lambda_s \cos \Lambda_a) \} + \dot{\theta}_y [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{ \omega'_k (\sin \psi_k \sin \Lambda_s + \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) + \\
& (\cos \psi_k \cos \theta_I \sin \Lambda_a) \} + \dot{\theta}_z a (\beta_d + \beta_p \cos \theta_I) - \dot{\theta}_z (e_1 + e_2 + \sum_{i=1}^{n-1} u_i) \sin \theta_I - \dot{\theta}_z (x_k + u_k) [\sin \Lambda_s (\beta_p + \beta_d \\
& \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I] + \dot{\theta}_z v'_k [\cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \sin \Lambda_s \cos \Lambda_a \sin \theta_I]
\end{aligned}$$

$$\begin{aligned}
& + \dot{\theta}_z \omega_k \sin \Lambda_a \sin \theta_I + \dot{\theta}_z [\eta \cos(\theta_G + \phi_k) - \xi \sin(\theta_G + \phi_k)] \{ v'_k [\sin \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \sin \theta_I \\
& \cos \Lambda_s \cos \Lambda_a] + \cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \sin \Lambda_s \cos \Lambda_a \sin \theta_I \} + \dot{\theta}_z [\eta \sin(\theta_G + \phi_k) + \xi \\
& \cos(\theta_G + \phi_k)] \{ \omega'_k [\sin \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I] + \sin \theta_I \sin \Lambda_a \} + \dot{\theta}_I a \cos \theta_I - \\
& \dot{\theta}_I e_1 (\beta_p \sin \theta_I) - \dot{\theta}_I \sum_{i=1}^{n-1} u_i (\beta_s \cos \theta_I + \beta_d \sin \theta_I) - \dot{\theta}_I (x_k + u_k) [\sin \Lambda_s + \cos \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] \\
& + \theta_I v_k [\cos \Lambda_s - \sin \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] + \theta_I \omega_k (\sin \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)) + \theta_I [\eta \cos(\theta_G + \phi_k) \\
& - \xi \sin(\theta_G + \phi_k)] \{ v'_k [\sin \Lambda_s + (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \cos \Lambda_s \cos \Lambda_a] + [\cos \Lambda_s - (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \sin \Lambda_s \\
& \cos \Lambda_a] \} + \dot{\theta}_I [\eta \sin(\theta_G + \phi_k) + \xi \cos(\theta_G + \phi_k)] \{ \omega'_k [\sin \Lambda_s + \cos \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] + [\sin \Lambda_a \\
& (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] \} \}
\end{aligned} \tag{4.23}$$

# Chapter 5

## Equation of Motion for Rotor Blade

The coupled equations of motion of the rotor blade can be derived using Hamilton's principle. The mathematical form of Hamilton's principle is stated as follows:

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W_e) dt = 0 \quad (5.1)$$

Where  $U$  is the strain energy;  $T$  is kinetic energy;  $W_e$  is the work done by the non-conservative external loads

In this chapter, the expressions for the variation of kinetic energy and strain energy of the rotor blade are derived

### 5.1 Kinetic Energy of Blade

The total energy of the beam,  $T$ , is defined as:

$$\begin{aligned} T &= \frac{1}{2} \sum_i \iiint_V \rho \vec{V} \cdot \vec{V} dVol \\ &= \frac{1}{2} \sum_i \int_0^{(l_e)_i} \iint_A \rho \vec{V} \cdot \vec{V} d\eta d\zeta dx \end{aligned} \quad (5.2)$$

Where  $\vec{V}$  is the velocity of an arbitrary point 'p' on the blade cross section with respect to the inertial reference frame. The variation of kinetic energy,  $\delta T$  can be written as:

$$\delta T = \sum_i \int_0^{(l_e)_i} \iint_A \rho \vec{V} \cdot \delta \vec{V} d\eta d\zeta dx \quad (5.3)$$

Substituting for the velocity  $\vec{V}$  from Eq. 4.17 and integrating  $\delta T$  by parts with respect to time, yields

$$\delta T = \sum_i \int_0^{(l_e)} \iint_A \rho [Z_u \delta u_k + Z_v \delta v_k + Z_w \delta w_k + Z'_v \delta v'_k + Z'_w \delta w'_k + Z_\phi \delta \phi_k] d\eta d\zeta dx \quad (5.4)$$

Where the terms  $Z_u, Z_v, Z_w, Z'_v, Z'_w$  and  $Z_\phi$  are the coefficients of  $\delta u_k, \delta v_k, \delta w_k, \delta v'_k, \delta w'_k$  and  $\delta \phi_k$  in the variation of kinetic energy expression.

Integration of the expression over the cross-section yields:

$$\delta T = m\Omega_0^2 l^3 \sum_i \int_0^{(l_e)} [\bar{Z}_u \delta u_k + \bar{Z}_v \delta v_k + \bar{Z}_w \delta w_k + \bar{Z}'_v \delta v'_k + \bar{Z}'_w \delta w'_k + \bar{Z}_\phi \delta \phi_k] dx \quad (5.5)$$

After eliminating the higher order terms, the expressions for  $\bar{Z}_u, \bar{Z}_v, \bar{Z}_w, \bar{Z}'_v, \bar{Z}'_w$  and  $\bar{Z}_\phi$  are obtained.

The cross-section integrals are defined as:

$$\begin{aligned} m &= \iint_A \rho d\eta d\zeta \\ m\eta_m &= \iint_A \rho \eta d\eta d\zeta \\ m\zeta_m &= \iint_A \rho \zeta d\eta d\zeta \\ Im_{\eta\eta} &= \iint_A \rho \eta^2 d\eta d\zeta \\ Im_{\zeta\zeta} &= \iint_A \rho \zeta^2 d\eta d\zeta \\ Im_{\eta\zeta} &= \iint_A \rho \eta \zeta d\eta d\zeta \end{aligned}$$

Where  $m$  is mass per unit length of the blade;  $m\eta_m$  and  $m\zeta_m$  are the first moments of cross-sectional mass per unit length;  $Im_{\eta\eta}, Im_{\zeta\zeta}$  and  $Im_{\eta\zeta}$  are the cross-sectional mass moments of inertia per unit length of the beam.

## 5.2 Strain Energy of Blade

The formulation of strain energy outlined in this section essentially follows the procedure given in the Refs. [6], and [5]

### 5.2.1 Strain Energy of the beam Element

The strain energy of the beam element is given by

$$U = \frac{1}{2} E_o I^3 \int_0^{l_e} \int \int \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{x\eta} \\ \gamma_{x\zeta} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} d\eta \, d\zeta \, dx \quad (5.6)$$

### 5.2.2 Explicit Strain-Displacement Relations

The expression for non zero strain components written in terms of  $u$ ,  $v$ ,  $w$  and  $\phi$  are given in Refs. [6], and [5]. They are:

$$\begin{aligned} \varepsilon_{xx} &= u_x + \frac{1}{2} v_x^2 + \frac{1}{2} w_x^2 + \frac{1}{2} (\eta^2 + \zeta^2) \phi_x^2 + \alpha_x \psi \\ &\quad + \alpha \tau_o (\zeta \psi_\eta - \eta \psi_\zeta) - [\eta \cos(\theta_g + \phi) - \zeta \sin(\theta_g + \phi)] v_{xx} \\ &\quad - [\eta \sin(\theta_g + \phi) + \zeta \cos(\theta_g + \phi)] w_{xx} + \eta (\bar{\gamma}_{x\eta,x} - \tau_o \bar{\gamma}_{x\zeta}) + \zeta (\bar{\gamma}_{x\zeta,x} + \tau_o \bar{\gamma}_{x\eta}) \\ \gamma_{x\eta} &= \bar{\gamma}_{x\eta} + \alpha \psi_\eta - \zeta (\phi_x + \phi_o) \\ \gamma_{x\zeta} &= \bar{\gamma}_{x\zeta} + \alpha \psi_\zeta - \eta (\phi_x + \phi_o) \end{aligned}$$

Where,

$$\phi_o = (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) (-v_x \sin \theta_G + w_x \cos \theta_G) \quad (5.7)$$

The underlined term in  $\varepsilon_{xx}$  represents the axial strain at the elastic axis. These strain expressions can be simplified using the following assumptions:

- The transverse shear at the elastic axis is assumed to be zero.
- The warping amplitude  $\alpha$  is assumed to be equal to  $-\phi_x$ .

The simplified strain components can be written as

$$\begin{aligned}\varepsilon_{xx} &= u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2 + \frac{1}{2}(\eta^2 + \zeta^2)\phi_x^2 - \underline{\psi\phi_{xx}} - [\tau_0(\zeta\psi_\eta - \eta\psi_\zeta)]\phi_x \\ &\quad - [\eta \cos(\theta_G + \phi_K) - \zeta \sin(\theta_G + \phi_K)]v_{xx} - [\eta \sin(\theta_G + \phi_K) + \zeta \cos(\theta_G + \phi_K)]w_{xx} \\ \gamma_{x\eta} &= -(\psi_\eta + \zeta)\phi_x - \zeta\phi_o \\ \gamma_{x\zeta} &= -(\psi_\zeta - \eta)\phi_x + \eta\phi_o\end{aligned}$$

where

$$\phi_o = (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G)(-v_x \sin \theta_G + w_x \cos \theta_G)$$

(5.8)

### 5.2.3 Stress-Strain Relations

Assuming that the blade is made of isotropic material, the stress-strain relationship is given by the following equations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{x\eta} \\ \gamma_{x\zeta} \end{Bmatrix} \quad (5.9)$$

### 5.2.4 Strain energy variation

The variation of strain energy of the  $i^{\text{th}}$  element is given by

$$\delta U = \frac{1}{2} E_o l_3 \int_0^{l_e} \int \int \left\{ \begin{array}{c} \delta \varepsilon_{xx} \\ \delta \gamma_{x\eta} \\ \delta \gamma_{x\zeta} \end{array} \right\}^T \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{array} \right\} d\eta d\zeta dx \quad (5.10)$$

The variation of the strain components are given as follows:

$$\begin{aligned} \delta \varepsilon_{xx} &= \delta u_x + v_x \delta v_x + w_x \delta w_x + (\eta^2 + \zeta^2) \phi_x \delta \phi_x - \psi \delta \phi_{xx} - [\tau_0 (\zeta \psi_\eta - \eta \psi_\zeta)] \delta \phi_x \\ &\quad - [\eta \cos(\theta_G + \phi_K) - \zeta \sin(\theta_G + \phi_K)] (\delta v_{xx} + \phi \delta w_{xx} + w_{xx} \delta \phi) \\ &\quad - [\eta \sin(\theta_G + \phi_K) + \zeta \cos(\theta_G + \phi_K)] (\delta w_{xx} - \phi \delta v_{xx} - v_{xx} \delta \phi) \\ \delta \gamma_{x\eta} &= -(\psi_\eta + \zeta) \delta \phi_x - \zeta \delta \phi_o \\ \gamma_{x\zeta} &= -(\psi_\zeta - \eta) \delta \phi_x + \eta \delta \phi_o \end{aligned}$$

where

$$\begin{aligned} \delta \phi_o &= (-\delta v_x \sin \theta_G + \delta w_x \cos \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \\ &\quad (-v_x \sin \theta_G + w_x \cos \theta_G) (\delta v_{xx} \cos \theta_G + \delta w_{xx} \sin \theta_G) \end{aligned} \quad (5.11)$$

It is assumed that the variations of strain components are of the same order as the corresponding strain components. Substituting the above expressions in the strain energy variation, we get the components of variation of strain energy as follows:

$$\begin{aligned} \delta U &= \int \int \int \left\{ \begin{array}{c} \delta u_x \left\{ E \left[ -\psi \phi_{xx} + u_x + 0.5 v_x^2 + 0.5 w_x^2 + 0.5 (\eta^2 + \zeta^2) \phi_x^2 - [\tau_0 (\zeta \psi_\eta - \eta \psi_\zeta)] \phi_x \right. \right. \\ \left. \left. - [\eta \cos(\theta_G + \phi_K) - \zeta \sin(\theta_G + \phi_K)] v_{xx} - [\eta \sin(\theta_G + \phi_K) + \zeta \cos(\theta_G + \phi_K)] w_{xx} \right] \right\} \\ + \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
& \delta v_x \{ -\eta G \sin \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [-\phi_x (-\eta + \psi_\zeta)] \\
& + (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) (-v_x \sin \theta_G + w_x \cos \theta_G) \eta \} \\
& + \zeta G \sin \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [-\phi_x (\psi_\eta + \zeta)] \\
& - (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) (-v_x \sin \theta_G + w_x \cos \theta_G) \zeta \} \\
& + \{ E v_x \left[ -\psi \phi_{xx} + u_x + 0.5 v_x^2 + 0.5 w_x^2 + 0.5 (\eta^2 + \zeta^2) \phi_x^2 - [\tau_0 (\zeta \psi_\eta - \eta \psi_\zeta)] \phi_x \right. \\
& \left. - [\eta \cos(\theta_G + \phi_K) - \zeta \sin(\theta_G + \phi_K)] v_{xx} - [\eta \sin(\theta_G + \phi_K) + \zeta \cos(\theta_G + \phi_K)] w_{xx} \right] \} \\
& + \\
& \delta v_{xx} \{ \eta G \cos \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) [-\phi_x (-\eta + \psi_\zeta)] \\
& + (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \} \\
& + \zeta G \cos \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) [-\phi_x (\psi_\eta + \zeta)] \\
& - (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \} \\
& + E \{ -\eta \cos(\theta_G + \phi) + \zeta \sin(\theta_G + \phi) [-\phi_{xx} \psi + u_x + 0.5 (v_x^2 + w_x^2) \\
& - \phi_x \tau_0 (\psi_\eta \zeta - \psi_\zeta \eta) + 0.5 \phi_x^2 (\eta^2 + \zeta^2) \\
& - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) \\
& - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi))] \} \\
& + \\
& \delta w_x \{ \eta G \cos \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [-\phi_x (-\eta + \psi_\zeta)] \\
& + \eta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \} \\
& - \zeta G \cos \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [-\phi_x (\psi_\eta + \zeta)] \\
& - \zeta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \} \\
& + E w_x \left[ -\psi \phi_{xx} + u_x + 0.5 (v_x^2 + w_x^2) + 0.5 (\eta^2 + \zeta^2) \phi_x^2 - [\tau_0 (\zeta \psi_\eta - \eta \psi_\zeta)] \phi_x \right. \\
& \left. - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) \right. \\
& \left. - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi)) \right\}
\end{aligned}$$

$$\begin{aligned}
& \delta w_{xx} \left\{ \eta G \sin \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) \left[ -\phi_x (-\eta + \psi_\zeta) \right. \right. \\
& + \eta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \left. \right] \\
& - \zeta G \cos \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) \left[ -\phi_x (\psi_\eta + \zeta) \right. \\
& - \zeta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \left. \right] \\
& - E (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) \left[ -\phi_{xx} \psi + u_x + 0.5 (v_x^2 + w_x^2) \right. \\
& - \phi_x \tau_0 (\psi_\eta \zeta - \psi_\zeta \eta) + 0.5 \phi_x^2 (\eta^2 + \zeta^2) \\
& - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) \\
& - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi)) \left. \right\} \\
& + \\
& \delta \phi \left\{ E [v_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) - w_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi))] \right. \\
& \left[ -\phi_{xx} \psi + u_x + 0.5 (v_x^2 + w_x^2) - \phi_x \tau_0 (\psi_\eta \zeta - \psi_\zeta \eta) + 0.5 \phi_x^2 (\eta^2 + \zeta^2) \right. \\
& - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi)) \left. \right\} \\
& + \\
& \delta \phi_x \left\{ G (\eta - \psi_\zeta) \left[ -\phi_x (-\eta + \psi_\zeta) + \eta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \right] \right. \\
& + G (-\psi_\eta - \zeta) \left[ -\phi_x (\psi_\eta + \zeta) - \zeta (w_x \cos \theta_G - v_x \sin \theta_G) (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \right] \\
& + E (-\tau_0 (\psi_\eta \zeta - \psi_\zeta \eta) + \phi_x (\eta^2 + \zeta^2)) \left[ -\phi_{xx} \psi + u_x + 0.5 (v_x^2 + w_x^2) \right. \\
& - \tau_0 \phi_x (\psi_\eta \zeta - \psi_\zeta \eta) + 0.5 \phi_x^2 (\eta^2 + \zeta^2) \\
& - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi)) \left. \right\} \\
& + \\
& \delta \phi_{xx} \left\{ -E \psi \left[ -\phi_{xx} \psi + u_x + 0.5 (v_x^2 + w_x^2) - \tau_0 \phi_x (\psi_\eta \zeta - \psi_\zeta \eta) + 0.5 \phi_x^2 (\eta^2 + \zeta^2) \right. \right. \\
& - w_{xx} (\zeta \cos(\theta_G + \phi) + \eta \sin(\theta_G + \phi)) - v_{xx} (\eta \cos(\theta_G + \phi) - \zeta \sin(\theta_G + \phi)) \left. \right\} \\
& \left. \right\} d\eta d\zeta dx \tag{5.12}
\end{aligned}$$

Now we define some cross-sectional integrals associated with the strain energy:

$$EA = \iint [E] d\eta d\zeta$$

$$EA\eta_a = \iint [E\eta] d\eta d\zeta$$

$$EA\zeta_0 = \iint [E\zeta] d\eta d\zeta$$

$$EI_{\eta\eta} = \iint [E\zeta^2] d\eta d\zeta$$

$$EI_{\zeta\zeta} = \iint [E\eta^2] d\eta d\zeta$$

$$EI_{\eta\zeta} = \iint [E\eta\zeta] d\eta d\zeta$$

$$EAC_0 = \iint [E(\eta^2 + \zeta^2)] d\eta d\zeta$$

$$EAC_1 = \iint [E\eta(\eta^2 + \zeta^2)] d\eta d\zeta$$

$$EAC_2 = \iint [E\zeta(\eta^2 + \zeta^2)] d\eta d\zeta$$

$$EAC_3 = \iint [E(\eta^2 + \zeta^2)^2] d\eta d\zeta$$

$$EAD_0 = \iint [E\psi] d\eta d\zeta$$

$$EAD_1 = \iint [E\eta\psi] d\eta d\zeta$$

$$EAD_2 = \iint [E\zeta\psi] d\eta d\zeta$$

$$EAD_3 = \iint [E\psi^2] d\eta d\zeta$$

$$EAD_4 = \iint [E\psi(\eta^2 + \zeta^2)] d\eta d\zeta$$

$$EAD_5 = \iint [E\psi(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$EAD'_0 = \iint [E(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$EAD'_1 = \iint [E\eta(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$EAD'_2 = \iint [E\zeta(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$EAD'_3 = \iint [E(\zeta\psi_\eta - \eta\psi_\zeta)^2] d\eta d\zeta$$

$$EAD'_4 = \iint [E(\eta^2 + \zeta^2)(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$GJ_0 = \iint [G(\eta^2 + \zeta^2)] d\eta d\zeta$$

$$GJ_1 = \iint [G(\zeta\psi_\eta - \eta\psi_\zeta)] d\eta d\zeta$$

$$GJ_2 = \iint [G(\psi_\eta^2 + \psi_\zeta^2)] d\eta d\zeta$$

Certain additional terms are defined as :

$$\overline{EA\eta_a} = EA\eta_a \cos\theta_G - EA\zeta_a \sin\theta_G$$

$$\overline{EA\zeta_a} = EA\eta_a \sin\theta_G + EA\zeta_a \cos\theta_G$$

$$\overline{EAC_1} = EAC_1 \cos\theta_G - EAC_2 \sin\theta_G$$

$$\overline{EAC_2} = EAC_1 \sin\theta_G + EAC_2 \cos\theta_G$$

$$\overline{EAD_1} = EAD_1 \cos\theta_G - EAD_2 \sin\theta_G$$

$$\overline{EAD_2} = EAD_1 \sin\theta_G + EAD_2 \cos\theta_G$$

$$\overline{EAD'_1} = EAD'_1 \cos\theta_G - EAD'_2 \sin\theta_G$$

$$\overline{EAD'_2} = EAD'_1 \sin\theta_G + EAD'_2 \cos\theta_G$$

$$\overline{EI_{\eta\zeta}} = EI_{\eta\zeta} \cos\theta_G - EI_{\eta\eta} \sin\theta_G$$

$$\overline{EI_{\eta\eta}} = EI_{\eta\zeta} \sin\theta_G + EI_{\eta\eta} \cos\theta_G$$

$$\overline{EI_{\zeta\zeta}} = EI_{\zeta\zeta} \cos\theta_G - EI_{\zeta\eta} \sin\theta_G$$

$$\overline{EI_{\zeta\eta}} = EI_{\zeta\zeta} \sin\theta_G + EI_{\zeta\eta} \cos\theta_G$$

$$GJ = GJ_0 + 2GJ_1 + GJ_2$$

After the integration over the area, we get the following line integral after separation of linear and nonlinear terms:

$$\begin{aligned}
U = & \int \left\{ \right. \\
& \delta u_x \left\{ EAu_x - \phi_{xx} EAD_0 - \phi_x \tau_0 EAD'_0 - w_{xx} \overline{EA\zeta_a} - v_{xx} \overline{EA\eta_a} \right\}_{linear} \\
& + \\
& \delta u_x \left\{ 0.5EA(v_x^2 + w_x^2) + 0.5\phi_x^2 EAC_0 - \phi \left( w_{xx} \overline{EA\eta_a} + v_{xx} \overline{EA\zeta_a} \right) \right\}_{nonlinear} \\
& + \\
& \delta v_x \left\{ v_x \left[ EA(u_x + 0.5(v_x^2 + w_x^2)) - \phi_{xx} EAD_0 + 0.5\phi_x^2 EAC_0 - \phi_x \tau_0 EAD'_0 \right. \right. \\
& \left. \left. - w_{xx} (\overline{EA\zeta_a} + \phi \overline{EA\eta_a}) - v_{xx} (\overline{EA\eta_a} - \phi \overline{EA\zeta_a}) \right] \right. \\
& \left. + \sin \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [\phi_x (-GJ_0 - GJ_1) - \phi_0 GJ_0] \right\}_{nonlinear} \\
& + \\
& \delta v_{xx} \left\{ \phi_{xx} \overline{EAD_1} - u_x \overline{EA\eta_a} + \phi_x \tau_0 \overline{EAD'_1} + w_{xx} (\overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G) \right. \\
& \left. + v_{xx} (\overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G) \right\}_{linear} \\
& + \\
& \delta v_{xx} \left\{ \cos \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) [\phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0] \right. \\
& \left. - 0.5(v_x^2 + w_x^2) \overline{EA\eta_a} - 0.5\phi_x^2 \overline{EAC_1} + \phi \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\zeta_a} \right. \right. \\
& \left. \left. - \phi_x \tau_0 \overline{EAD'_2} + 0.5\phi_x^2 \overline{EAC_2} - w_{xx} \left[ (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G + (\overline{EI_{\eta\eta}} - \overline{EI_{\zeta\zeta}}) \cos \theta_G \right] \right. \right. \\
& \left. \left. - v_{xx} \left[ (\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \sin \theta_G + (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \cos \theta_G \right] \right\}_{nonlinear} \right. \\
& + \\
& \delta w_x \left\{ w_x \left[ EA(u_x + 0.5(v_x^2 + w_x^2)) - \phi_{xx} EAD_0 + 0.5\phi_x^2 EAC_0 - \phi_x \tau_0 EAD'_0 \right. \right. \\
& \left. \left. - w_{xx} (\overline{EA\zeta_a} + \phi \overline{EA\eta_a}) - v_{xx} (\overline{EA\eta_a} - \phi \overline{EA\zeta_a}) \right] \right. \\
& \left. + \cos \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) [\phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0] \right\}_{nonlinear} \\
& + \\
& \delta w_{xx} \left\{ \phi_{xx} \overline{EAD_2} - u_x \overline{EA\zeta_a} + \phi_x \tau_0 \overline{EAD'_2} + w_{xx} (\overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G) \right. \\
& \left. + v_{xx} (\overline{EI_{\eta\zeta}} \cos \theta_G + \overline{EI_{\zeta\zeta}} \sin \theta_G) \right\}_{linear} \\
& +
\end{aligned}$$

$$\begin{aligned}
& \delta w_{xx} \left\{ \sin \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) \left[ \phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0 \right] \right. \\
& - 0.5 (v_x^2 + w_x^2) \overline{EA\zeta_a} - 0.5 \phi_x^2 \overline{EAC_2} + \phi \left[ -\phi_{xx} \overline{EAD_1} + (u_{xx} + 0.5 (v_x^2 + w_x^2)) \overline{EA\eta_a} \right. \\
& - \phi_x \tau_0 \overline{EAD'_1} + 0.5 \phi_x^2 \overline{EAC_1} - w_{xx} \left[ (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \cos \theta_G + (\overline{-EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G \right] \\
& \left. \left. - v_{xx} \left[ (\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G \right] \right\}_{nonlinear} \right. \\
& + \\
& \delta \phi \left\{ v_{xx} \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5 (v_x^2 + w_x^2)) \overline{EA\zeta_a} - \phi_x \tau_0 \overline{EAD'_2} + 0.5 \phi_x^2 \overline{EAC_2} \right. \right. \\
& - w_{xx} \left[ \overline{EI_{\zeta\eta}} \sin \theta_G + \overline{EI_{\eta\eta}} \cos \theta_G \right] - v_{xx} \left[ \overline{EI_{\zeta\zeta}} \sin \theta_G + \overline{EI_{\eta\zeta}} \cos \theta_G \right] \\
& + \phi \left[ -\phi_{xx} \overline{EAD_1} + (u_x + 0.5 (v_x^2 + w_x^2)) \overline{EA\eta_a} \right] - \phi_x \tau_0 \overline{EAD'_1} + 0.5 \phi_x^2 \overline{EAC_1} \\
& - w_{xx} [(\overline{EI_{\zeta\eta}} + \overline{EI_{\zeta\eta}}) \cos \theta_G + (\overline{-EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G] \\
& - v_{xx} [(\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G] \\
& + w_{xx} [-\phi_{xx} \overline{EAD_1} + (u_x + 0.5 (v_x^2 + w_x^2)) \overline{EA\eta_a} - \phi_x \tau_0 \overline{EAD'_1} + 0.5 \overline{EAC_1} \phi_x^2] \\
& + w_{xx} \left[ \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right] + v_{xx} \left[ \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\zeta\eta}} \sin \theta_G \right] \\
& + \phi \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5 (v_x^2 + w_x^2)) \overline{EA\zeta_a} - \phi_x \tau_0 \overline{EAD'_2} + 0.5 \phi_x^2 \overline{EAC_2} \right. \\
& \left. \left. - w_{xx} \left[ (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \cos \theta_G + (\overline{-EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G \right] \right. \right. \\
& \left. \left. - v_{xx} \left[ (\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G \right] \right\}_{nonlinear} \right. \\
& + \\
& \delta \phi_x \left\{ -GJ \phi_x - \tau_0 \left[ -\phi_{xx} EAD_5 + EAD'_0 u_x - \phi_x \tau_0 EAD'_3 - w_{xx} \overline{EAD'_1} - v_{xx} \overline{EAD'_1} \right] \right\}_{linear} \\
& + \\
& \delta \phi_x \left\{ -\tau_0 \left[ 0.5 EAD'_0 (v_x^2 + w_x^2) + 0.5 EAD'_4 \phi_x^2 - \phi (w_{xx} \overline{EAD'_1} - v_{xx} \overline{EAD'_2}) \right] \right. \\
& + \phi_x \left[ -\phi_{xx} EAD_4 + (u_x + 0.5 (v_x^2 + w_x^2)) EAC_0 - \phi_x \tau_0 EAD'_4 + 0.5 \phi_x^2 EAC_3 \right. \\
& \left. \left. - w_{xx} (\overline{EAC_2} + \phi \overline{EAC_1}) - v_{xx} (\overline{EAC_1} + \phi \overline{EAC_2}) \right] \right\}_{nonlinear} \\
& + \\
& \delta \phi_{xx} \left\{ \phi_{xx} EAD_3 - EAD_0 u_x + \phi_x \tau_0 EAD_5 + w_{xx} \overline{EAD_2} + v_{xx} \overline{EAD_1} \right\}_{linear} \\
& +
\end{aligned}$$

$$\begin{aligned}
& \delta\phi_{xx} \left\{ -0.5EAD_0(v_x^2 + w_x^2) - 0.5EAD_4\phi_x^2 + \phi \left( \underline{w_{xx}EAD_1} - \underline{v_{xx}EAD_2} \right) \right\}_{nonlinear} \\
& \} dx
\end{aligned} \tag{5.13}$$

The underlined terms are those, which can be linearized by replacing the extensional strain term by the axial inertial force. This will be shown in the later part of the report.

# Chapter 6

## Formulation of element matrices associated with kinetic energy variation

### 6.1 Finite element discretization

The variational expressions associated with the kinetic and potential energy of the rotor blades is nonlinear. The unknowns are the deformation functions  $u$ ,  $v$ ,  $w$  and  $\phi$ . These are dependent on both space and time. The spatial dependence is eliminated using a Rayleigh Ritz finite element formulation. The blade is divided into sub-regions (finite elements), shown in Fig. 6.1, and the total dynamic potential is calculated for each sub-region. By applying Hamilton's principle to each sub-region, a discretized form of the equations of motion can be obtained. In this development, each sub-region is modeled by a straight beam type finite element. These beam elements are located along the elastic axis (line of shear centers) of the blade.

The discretized form of Hamilton's principle is written as:

$$\int_{t_1}^{t_2} \sum_{i=1}^N (\delta U_i - \delta T_i - \delta W_{ei}) dt = 0$$

Where,

$N$  = Total number of finite elements in the model

$\delta U_i$  = Variation of the strain energy in the  $i^{th}$  element

$\delta T_i$  = Variation of the kinetic energy in the  $i^{th}$  element

$\delta W_{ei}$  = Virtual work of the external loads in  $i^{th}$  element

The degrees of freedom for the beam finite element are discretized in space and time in the following manner

$$\begin{bmatrix} v \\ w \\ \phi \\ u \end{bmatrix} = \begin{bmatrix} \{\phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\phi_u\}^T \end{bmatrix} \begin{bmatrix} \{V\} \\ \{W\} \\ \{\Phi\} \\ \{U\} \end{bmatrix}$$

where,  $\{\phi_v\}^T, \{\phi_w\}^T, \{\phi_\phi\}^T, \{\phi_u\}^T$  are space dependent interpolation functions; and  $\{V\}, \{W\}, \{\Phi\}, \{U\}$  are time dependent nodal values of  $v, w, \phi, u$  respectively.

$$\{V\} = \begin{bmatrix} v_1 \\ v'_1 \\ v_2 \\ v'_2 \end{bmatrix}, \{W\} = \begin{bmatrix} w_1 \\ w'_1 \\ w_2 \\ w'_2 \end{bmatrix}; \{\Phi\} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}; \{U\} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The nodal coordinates are shown in Fig. 6.2

The variation of the displacement functions for the beam can be written as:

$$\begin{bmatrix} \delta v \\ \delta w \\ \delta \phi \\ \delta u \end{bmatrix} = \begin{bmatrix} \{\phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\phi_u\}^T \end{bmatrix} \begin{bmatrix} \{\delta V\} \\ \{\delta W\} \\ \{\delta \Phi\} \\ \{\delta U\} \end{bmatrix}$$

In this development, Hermite interpolation polynomials are used for the space dependent interpolation functions. A cubic Hermite interpolation polynomial,  $\{\Phi_C\}$ , is used for the bending deflections ( $v, w$ ) and a quadratic interpolation functions,  $\{\Phi_q\}$ , is used for the torsional rotation ( $\phi$ ), and the axial deflection ( $u$ ). The mathematical expressions for these polynomials are given as:

$$\{\phi_v\} = \{\phi_w\} = \{\Phi_c\} = \begin{bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ l_e(\xi - 2\xi^2 + \xi^3) \\ 3\xi^2 - 2\xi^3 \\ l_e(-\xi^2 + \xi^3) \end{bmatrix} \quad (6.4)$$

$$\{\phi_\phi\} = \{\phi_u\} = \{\Phi_q\} = \begin{bmatrix} 1 - 3\xi + 2\xi^2 \\ 4\xi - 4\xi^2 \\ -\xi + 2\xi^2 \end{bmatrix} \quad (6.5)$$

Where:

$$\xi = \frac{x_k}{l_e}$$

$x_k$  = Span-wise (axial) coordinate of the beam element.

$l_e$  = Length of beam element.

For bending deformations, the nodal parameters are the displacements and slopes at both ends of the beam element. Therefore, the resulting elements will have inter-element continuity for both displacements and slopes. In addition, because of the cubic Hermite interpolation polynomial, bending strains vary linearly over the element length. The quadratic interpolation functions are used for torsional rotation ( $\phi$ ) and the deflection (u). This polynomial has the capability of modeling a linear variation of strains along the element length and therefore provides the same level of accuracy as the beam-bending element. The nodal parameters for these elements are chosen as the values of the displacements function at the two end nodes and at the mid-point of the element.

The resulting beam element has 14 degree of freedom: 4 in-plane (lag) bending degrees of freedom, 4 out-plane (flap) degrees of freedom, and 3 degrees of freedom each of torsion ( $\phi$ ), and axial deflection (u). The nodal degrees of freedom are shown in Fig.

6.2

## 6.2 Element matrices associated with kinetic energy variation

The beam element matrices associated with the kinetic energy variation are derived by substituting the assumed expressions for the displacements functions (eq. 6.2) in the kinetic energy variation  $\delta T$  (eq. 5.5) and carrying out the integration over the length of the beam element. The resulting variation of the kinetic energy can be written in the form:

$$\begin{aligned} \delta T_i = & -\{\delta q\}^T \left( [M]_{14 \times 14} \{\ddot{q}\} + [M^C]_{14 \times 14} \{\dot{q}\} + [K^{CF}]_{14 \times 14} \{q\} + [M^1]_{14 \times 3} \begin{Bmatrix} \ddot{R}_x \\ \ddot{R}_y \\ \ddot{R}_z \end{Bmatrix} + [M^2]_{14 \times 3} \begin{Bmatrix} \dot{R}_x \\ \dot{R}_y \\ \dot{R}_z \end{Bmatrix} \right. \\ & \left. + [M^3]_{14 \times 3} \begin{Bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{Bmatrix} + [M^4]_{14 \times 3} \begin{Bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix} + \{V^I\}_{14 \times 1} + \{V^L\}_{14 \times 1} + \{V^{NL}\}_{14 \times 1} \right) \end{aligned} \quad (6.6)$$

Where  $\{q\}$  represents the vector of unknown nodal degrees of freedom

$$\{q\}_{14 \times 1} = \begin{Bmatrix} \{V\} \\ \{W\} \\ \{\Phi\} \\ \{U\} \end{Bmatrix} \quad (6.7)$$

Detailed expressions for the various matrices defined in Eq. 6.6 are given as follows:

### 6.2.1 Mass Matrix $[M]_{14 \times 14}$ :

$$[M_{11}] = - \int_0^L \left[ m \{ \Phi_c \} \{ \Phi_q \}^T + \{ m \eta_m \sin(\theta_G + \Phi_K) + m \zeta_m \cos(\theta_G + \Phi_K) \} \{ \Phi_c \} \{ \Phi_q \}^T \right]$$

$$- 2 \operatorname{Im}_{\eta\zeta} \sin(\theta_G + \Phi_K) \cos(\theta_G + \Phi_K) \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx$$

$$[M_{12}] = \int_0^L \left[ \{ (\operatorname{Im}_{\eta\eta} - \operatorname{Im}_{\zeta\zeta}) \sin(\theta_G + \Phi_K) \cos(\theta_G + \Phi_K) \} + \operatorname{Im}_{\eta\zeta} [\cos^2(\theta_G + \Phi_K) \right.$$

$$\left. - \sin^2(\theta_G + \Phi_K) \} \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx$$

$$[M_{13}] = - \int_0^L \left[ \{ m \eta_m \sin(\theta_G + \Phi_K) + m \zeta_m \cos(\theta_G + \Phi_K) \} \{ \Phi_c \} \{ \Phi_q \}^T \right] dx$$

$$[M_{14}] = - \int_0^L \left[ \{ m \eta_m \cos(\theta_G + \Phi_K) - m \zeta_m \sin(\theta_G + \Phi_K) \} \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx$$

$$[M_{21}] = \int_0^L \left[ \{ (\operatorname{Im}_{\eta\eta} - \operatorname{Im}_{\zeta\zeta}) \sin(\theta_G + \Phi_K) \cos(\theta_G + \Phi_K) + \operatorname{Im}_{\eta\zeta} [\cos^2(\theta_G + \Phi_K) - \right.$$

$$\left. \sin^2(\theta_G + \Phi_K) \} \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx = [M_{12}]^T$$

$$[M_{22}] = \int_0^L \left[ m \{ \Phi_c \} \{ \Phi_q \}^T + \{ \operatorname{Im}_{\eta\eta} \cos^2(\theta_G + \Phi_K) + \operatorname{Im}_{\zeta\zeta} \sin^2(\theta_G + \Phi_K) + \right.$$

$$\left. 2 \operatorname{Im}_{\eta\zeta} \sin(\theta_G + \Phi_K) \cos(\theta_G + \Phi_K) \} \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx$$

$$[M_{23}] = \int_0^L \left[ \{ m \eta_m \cos(\theta_G + \Phi_K) - m \zeta_m \sin(\theta_G + \Phi_K) \} \{ \Phi_c \} \{ \Phi_q \}^T \right] dx$$

$$[M_{24}] = - \int_0^L \left[ \{ m \eta_m \sin(\theta_G + \Phi_K) + m \zeta_m \cos(\theta_G + \Phi_K) \} \{ \Phi'_c \} \{ \Phi'_q \}^T \right] dx$$

$$[M_{23}] = [M_{13}]^T$$

$$[M_{32}] = [M_{23}]^T$$

$$[M_{33}] = \int_0^{l_e} \left\{ \text{Im}_{\eta\eta} + \text{Im}_{\zeta\zeta} \right\} \{\Phi_q\} \{\Phi_q\}^T dx$$

$$[M_{34}] = [0]$$

$$[M_{41}] = [M_{14}]^T$$

$$[M_{42}] = [M_{24}]^T$$

$$[M_{43}] = [0]$$

$$[M_{44}] = \int_0^{l_e} m \{\Phi_q\} \{\Phi_q\}^T dx$$

## 6.2.2 Damped Mass Matrix $[M^c]_{14 \times 14}$ :

$$[M_{11}^c] = \int_0^{l_e} m [-\sin \Lambda_s \cos \Lambda_a \cos \Lambda_s - \cos \Lambda_s \sin \Lambda_s \cos \Lambda_a + \sin \Lambda_s \sin \Lambda_a \cos \Lambda_s] \{\Phi_c\} \{\Phi_c\}^T$$

$$+ [-m\eta_m \cos(\theta_G + \phi_k) + m\zeta_m \sin(\theta_G + \phi_k)] \{\cos \Lambda_s \cos \Lambda_a + \sin \Lambda_s \sin \Lambda_a \cos \Lambda_a\} \{\Phi_c\} \{\Phi_c'\}^T dx$$

$$[M_{12}^c] = \int_0^{l_e} m \cos \Lambda_s \cos \Lambda_a \{\Phi_c\} \{\Phi_c\}^T dx$$

$$[M_{13}^c] = \int_0^{l_e} \left[ 2 \cos \Lambda_s \cos \Lambda_a (I_{\eta\eta} + I_{\zeta\zeta}) \sin(\theta_G + \phi_k) \cos(\theta_G + \phi_k) \{\Phi_c'\} \{\Phi_q\}^T \right. \\ \left. - \cos \Lambda_a \cos \Lambda_s I_{\eta\eta} (\cos^2(\theta_G + \phi_k) - \sin^2(\theta_G + \phi_k)) \right]$$

$$[M_{14}^c] = 0$$

$$[M_{21}^c] = \int_0^{l_e} \left[ \cos \Lambda_s \cos \Lambda_a \left\{ (m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)) \right. \right. \\ \left. \left. - (m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)) \right\} ((\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1)) \{\Phi_c\} \{\Phi_c'\}^T \right. \\ \left. + \cos \Lambda_a \cos \Lambda_s \left\{ m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k) \right\} \{\Phi_c\} \{\Phi_c'\}^T \right] dx$$

$$[M_{22}^c] = \int_0^{l_e} \left[ \cos \Lambda_s \cos \Lambda_a \left\{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \right\} \right. \\ \left. ((\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1)) \{\Phi_c\} \{\Phi_c\}^T \right] dx$$

$$[M_{23}^c] = \int_0^{l_e} \left[ - \left\{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \right\} \cos \Lambda_a \{\Phi_c\} \{\Phi_q\}^T \right. \\ \left. - \sin \Lambda_s \sin \Lambda_a \left\{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \right\} \{\Phi_c'\} \{\Phi_q\}^T \right. \\ \left. - \cos \Lambda_s \sin^2 \Lambda_a \left\{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \right\} \{\Phi_c'\} \{\Phi_q\}^T \right. \\ \left. - \cos \Lambda_s \cos \Lambda_a (I_{\zeta\zeta} - I_{\eta\eta}) (\sin^2(\theta_G + \phi_k) - \cos^2(\theta_G + \phi_k)) \{\Phi_c'\} \{\Phi_q\}^T \right]$$

$$\begin{aligned}
& + \left\{ (I_{\zeta\zeta} - I_{\eta\eta} - 2I_{\eta\zeta}) \right\} \sin(\theta_G + \phi_k) \cos(\theta_G + \phi_k) \cos^2 \Lambda_s \cos^2 \Lambda_a \left\{ \Phi'_c \right\} \left\{ \Phi_q \right\}^T \Big] dx \\
\left[ M_{24}^c \right] &= 0 \\
\left[ M_{31}^c \right] &= \int_0^{l_e} \left( \cos \Lambda_s \cos \Lambda_a \left[ \left\{ I_{\zeta\zeta} \cos(\theta_G + \phi_k) - I_{\eta\zeta} \sin(\theta_G + \phi_k) \right\} \right. \right. \\
& \left. \left. - \sin(\theta_G + \phi_k) \cos \Lambda_s + \cos(\theta_G + \phi_k) \sin \Lambda_s \sin \Lambda_a \right] + \left\{ -I_{\eta\zeta} \cos(\theta_G + \phi_k) \right. \right. \\
& \left. \left. + I_{\eta\eta} \sin(\theta_G + \phi_k) \right\} \left( -\cos \Lambda_s \cos(\theta_G + \phi_k) - \sin(\theta_G + \phi_k) \sin \Lambda_s \sin \Lambda_a \right) \right] \\
& - \cos^2 \Lambda_s \cos \Lambda_a \left[ \left\{ I_{\zeta\zeta} - I_{\eta\eta} \right\} \cos(\theta_G + \phi_k) \sin(\theta_G + \phi_k) \right. \\
& \left. + I_{\eta\zeta} \left( \cos^2(\theta_G + \phi_k) - \sin^2(\theta_G + \phi_k) \right) \right] \left\{ \Phi_q \right\} \left\{ \Phi'_c \right\}^T dx \\
\left[ M_{32}^c \right] &= \int -\sin \Lambda_a \left[ \left\{ I_{\zeta\zeta} \sin(\theta_G + \phi_k) + I_{\eta\zeta} \cos(\theta_G + \phi_k) \right\} \left( \sin(\theta_G + \phi_k) \sin \Lambda_s \right. \right. \\
& \left. + \cos \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) \right) + \left\{ I_{\eta\zeta} \sin(\theta_G + \phi_k) + I_{\eta\eta} \cos(\theta_G + \phi_k) \right\} \\
& \left( \cos(\theta_G + \phi_k) \sin \Lambda_s - \cos \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) \right) \left. \right] - \left( \cos^2 \Lambda_s \cos \Lambda_a \right. \\
& \left. + \sin^2 \Lambda_s \cos \Lambda_a \right) \left\{ I_{\zeta\zeta} \sin^2(\theta_G + \phi_k) + I_{\eta\eta} \cos^2(\theta_G + \phi_k) \right. \\
& \left. + 2I_{\eta\zeta} \sin(\theta_G + \phi_k) \cos(\theta_G + \phi_k) \right\} - \cos \Lambda_s \cos \Lambda_a \left[ \left\{ I_{\zeta\zeta} \sin(\theta_G + \phi_k) \right. \right. \\
& \left. + I_{\eta\zeta} \cos(\theta_G + \phi_k) \right\} \left( -\cos \Lambda_s \sin(\theta_G + \phi_k) + \sin \Lambda_s \sin \Lambda_a \cos(\theta_G + \phi_k) \right) \\
& \left. + \left\{ I_{\eta\eta} \cos(\theta_G + \phi_k) + I_{\eta\zeta} \sin(\theta_G + \phi_k) \right\} \left( \cos \Lambda_s \cos(\theta_G + \phi_k) \right. \right. \\
& \left. \left. - \sin \Lambda_s \sin \Lambda_a \sin(\theta_G + \phi_k) \right) \right] \left\{ \Phi_c \right\} \left\{ \Phi'_c \right\}^T dx \\
\left[ M_{33}^c \right] &= \int_0^{l_e} \cos \Lambda_s \left[ \left\{ I_{\zeta\zeta} \sin(\theta_G + \phi_k) + I_{\eta\zeta} \cos(\theta_G + \phi_k) \right\} \right. \\
& \left( \sin(\theta_G + \phi_k) \sin \Lambda_s + \cos(\theta_G + \phi_k) \cos \Lambda_s \sin \Lambda_a \right) + \left\{ I_{\eta\zeta} \sin(\theta_G + \phi_k) \right. \\
& \left. + I_{\eta\eta} \cos(\theta_G + \phi_k) \right\} \left( \cos(\theta_G + \phi_k) \sin \Lambda_s - \sin(\theta_G + \phi_k) \cos \Lambda_s \sin \Lambda_a \right) \left\{ \Phi_q \right\} \left\{ \Phi_q \right\}^T dx \\
\left[ M_{34}^c \right] &= \int_0^{l_e} \left[ -m \cos \Lambda_s \cos \Lambda_a + \left\{ m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k) \right\} \right. \\
& \left( \cos \Lambda_s - \sin \Lambda_s \sin \Lambda_a \right) - \left\{ m\eta_m \sin(\theta_G + \phi_k) - m\zeta_m \cos(\theta_G + \phi_k) \right\} \\
& \left. \cos \Lambda_s \cos \Lambda_a \right] \left\{ \Phi_q \right\} \left\{ \Phi_q \right\}^T dx \\
\left[ M_{41}^c \right] &= \int_0^{l_e} m \left[ \cos^2 \Lambda_s \cos \Lambda_a + \sin^2 \Lambda_s \cos \Lambda_a - \cos^2 \Lambda_s \sin \Lambda_a \right] \left\{ \Phi_q \right\} \left\{ \Phi_c \right\}^T dx \\
\left[ M_{43}^c \right] &= \int_0^{l_e} -\cos \Lambda_s \cos \Lambda_a \left( m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k) \right) \left\{ \Phi_q \right\} \left\{ \Phi'_c \right\}^T dx \\
\left[ M_{44}^c \right] &= \int_0^{l_e} m \left[ -\cos^2 \Lambda_s \sin \Lambda_a \cos \Lambda_a + \sin \Lambda_s \sin \Lambda_a \cos \Lambda_s \right] \left\{ \Phi_q \right\} \left\{ \Phi_q \right\}^T dx
\end{aligned}$$

### 6.2.3 Centrifugal Stiffness Matrix $[K]_{14 \times 14}$ :

$$[K_{11}] = \int_0^{l_e} \left[ -m \left( \cos^2 \Lambda_a \cos^2 \theta_I + \cos^2 \theta_I \cos^2 \Lambda_s \sin^2 \Lambda_a \right. \right. \\ \left. \left. - \cos \theta_I \sin^2 \Lambda_s \sin \Lambda_a \cos \Lambda_a \left( \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \right) \right) \right. \\ \left. \left\{ \Phi_c \right\} \left\{ \Phi'_c \right\}^T + \left( \cos^2 \theta_I \sin^2 \Lambda_s \cos^2 \Lambda_a + \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a \right) \right. \\ \left. \left( \left\{ \text{Im}_{\zeta\zeta} \cos^2 (\theta_G + \phi_k) + \text{Im}_{\eta\eta} \sin^2 (\theta_G + \phi_k) \right\} - \left\{ I_{\eta\zeta} \sin 2(\theta_G + \phi_k) \right\} \right) \left\{ \Phi'_c \right\} \left\{ \Phi'_c \right\}^T \right] dx$$

$$[K_{12}] = \int_0^{l_e} \left[ m \left( \cos^2 \theta_I \sin \Lambda_s \sin \Lambda_a \cos \Lambda_a - \cos \theta_I \sin \Lambda_s \cos^2 \Lambda_a \right. \right. \\ \left. \left( \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \right) \right) \left\{ \Phi_c \right\} \left\{ \Phi'_c \right\}^T + \left( \cos^2 \theta_I \sin \Lambda_a \cos \Lambda_s \cos \Lambda_a \right. \\ \left. \left\{ m\eta_m \cos (\theta_G + \phi_k) - m\zeta_m \sin (\theta_G + \phi_k) \right\} \left\{ \Phi'_c \right\} \left\{ \Phi_c \right\}^T \right. \\ \left. - \left( \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a \right) \left( I_{\eta\eta} - I_{\zeta\zeta} \right) \sin (\theta_G + \phi_k) \cos (\theta_G + \phi_k) \right. \\ \left. - I_{\eta\zeta} \left\{ \cos 2(\theta_G + \phi_k) \right\} \right) \left\{ \Phi'_c \right\} \left\{ \Phi'_c \right\}^T \right] dx$$

$$[K_{13}] = 0$$

$$[K_{14}] = \int_0^{l_e} \left[ m \left( \cos^2 \theta_I \sin \Lambda_s \cos \Lambda_s \sin^2 \Lambda_a \right) \left\{ \Phi_c \right\} \left\{ \Phi_q \right\}^T \right. \\ \left. + \left( \cos^2 \theta_I \sin^2 \Lambda_s \cos \Lambda_a - \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a \right) \right. \\ \left. + \left\{ m\eta_m \cos (\theta_G + \phi_k) - m\zeta_m \sin (\theta_G + \phi_k) \right\} \left\{ \Phi'_c \right\} \left\{ \Phi_q \right\}^T \right] dx$$

$$[K_{21}] = [K_{12}]^T$$

$$[K_{22}] = \int_0^{l_e} \left[ -m \left( \cos^2 \theta_I \sin^2 \Lambda_a \sin^2 \Lambda_s + \cos^2 \theta_I \sin \Lambda_a \sin \Lambda_s \right) \left\{ \Phi_c \right\} \left\{ \Phi_c \right\}^T \right. \\ \left. + \left( \cos^2 \theta_I \sin^2 \Lambda_s \cos^2 \Lambda_a + \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_s \right) \right. \\ \left. \left( \left\{ \text{Im}_{\zeta\zeta} \sin^2 (\theta_G + \phi_k) + \text{Im}_{\eta\eta} \cos^2 (\theta_G + \phi_k) \right\} + \left\{ I_{\eta\zeta} \sin 2(\theta_G + \phi_k) \right\} \right) \left\{ \Phi'_c \right\} \left\{ \Phi'_c \right\}^T \right] dx$$

$$[K_{23}] = 0$$

$$[K_{24}] = \int_0^{l_e} \left[ m \left( \cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a \sin \Lambda_a - \cos \theta_I \cos \Lambda_s \cos^2 \Lambda_a \left( \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \right) \right) \right. \\ \left. \left\{ \Phi_c \right\} \left\{ \Phi_q \right\}^T + \left( \cos^2 \theta_I \sin^2 \Lambda_s \cos \Lambda_a - \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a - \cos \theta_I \cos^2 \Lambda_s \sin \Lambda_a \cos \Lambda_a \right. \right. \\ \left. \left. \left( \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \right) \right) \left\{ m\eta_m \sin (\theta_G + \phi_k) + m\zeta_m \cos (\theta_G + \phi_k) \right\} \left\{ \Phi'_c \right\} \left\{ \Phi_q \right\}^T \right] dx$$

$$[K_{31}] = \int_0^{l_e} \left[ \left\{ m\eta_m \sin (\theta_G + \phi_k) + m\zeta_m \cos (\theta_G + \phi_k) \right\} \left( \cos^2 \theta_I \cos^2 \Lambda_s + \alpha \cos \theta_I \right. \right. \\ \left. \left. + (e_1 + e_2) \left\{ \beta_s \cos \theta_I + \beta_d \sin \theta_I \right\} - e_1 \beta_p \sin \theta_I + \cos \theta_I \cos \Lambda_s \cos \Lambda_a \left\{ -\alpha (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right. \right. \right. \\ \left. \left. \left. + e_1 + e_2 + \sum_{i=1}^{n-1} I_{ei} + x_k \right\} \right) \left\{ \Phi_q \right\} \left\{ \Phi_c \right\}^T - \left[ \left( 2 \cos^2 \theta_I \sin \Lambda_s \cos \Lambda_s \cos \Lambda_a \right) \left( I_{\eta\eta} - I_{\zeta\zeta} \right) \right. \right. \\ \left. \left. \sin (\theta_G + \phi_k) \cos (\theta_G + \phi_k) + I_{\eta\zeta} \left\{ \cos^2 (\theta_G + \phi_k) - \sin^2 (\theta_G + \phi_k) \right\} \right) - \left( \left\{ \text{Im}_{\zeta\zeta} \cos^2 (\theta_G + \phi_k) \right. \right. \right. \\ \left. \left. \left. + \text{Im}_{\eta\eta} \sin^2 (\theta_G + \phi_k) \right\} + I_{\eta\zeta} \left\{ \cos 2(\theta_G + \phi_k) \right\} \right) \right] dx$$

$$+ \text{Im}_{\eta\eta} \sin^2(\theta_G + \phi_k) \} + \{ 2I_{\eta\zeta} \cos(\theta_G + \phi_k) \sin(\theta_G + \phi_k) \} \) (\cos^2 \theta_I \cos^2 \Lambda_s \sin \Lambda_a \cos \Lambda_a \\ - \cos^2 \theta_I \cos^2 \Lambda_a \cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) \] \{ \Phi_q \} \{ \Phi'_c \}^T \] dx$$

$$[K_{32}] = \int_0^{l_e} \left[ \left( \cos^2 \theta_I \sin \Lambda_s \sin \Lambda_a \{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \} \right. \right. \\ + \cos^2 \theta_I \sin \Lambda_s \cos \Lambda_s \sin \Lambda_a \{ m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k) \} \} \{ \Phi_q \} \{ \Phi_c \}^T \\ - \left[ (\cos \theta_I \sin \Lambda_s \cos^2 \Lambda_s \cos \Lambda_a) \{ (I_{\eta\eta} - I_{\zeta\zeta}) \sin(\theta_G + \phi_k) \cos(\theta_G + \phi_k) \right. \\ + I_{\eta\zeta} \{ \cos^2(\theta_G + \phi_k) - \sin^2(\theta_G + \phi_k) \} + \{ \text{Im}_{\zeta\zeta} \cos^2(\theta_G + \phi_k) + \text{Im}_{\eta\eta} \sin^2(\theta_G + \phi_k) \} \\ \left. \left. + \{ 2I_{\eta\zeta} \cos(\theta_G + \phi_k) \sin(\theta_G + \phi_k) \} \right) (\cos^2 \theta_I \cos \Lambda_s \sin \Lambda_s \cos \Lambda_a \right. \\ \left. - \cos^2 \theta_I \cos^2 \Lambda_a \cos \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) \right] \{ \Phi_q \} \{ \Phi'_c \}^T \] dx$$

$$[K_{33}] = \int_0^{l_e} \left[ (\cos^2 \theta_I \cos^2 \Lambda_a - \cos^2 \theta_I \sin^2 \Lambda_s \sin^2 \Lambda_a) \right. \\ \left. (\{ \text{Im}_{\zeta\zeta} \cos^2(\theta_G + \phi_k) + \text{Im}_{\eta\eta} \sin^2(\theta_G + \phi_k) \} - \{ \text{Im}_{\zeta\zeta} \sin^2(\theta_G + \phi_k) + \text{Im}_{\eta\eta} \cos^2(\theta_G + \phi_k) \}) \right. \\ \left. \{ \Phi_q \} \{ \Phi_q \}^T \right] dx$$

$$[K_{34}] = \int_0^{l_e} \left[ (\cos^2 \theta_I \sin \Lambda_s \cos \Lambda_s + \cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a \sin \Lambda_s) \right.$$

$$\left. \{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \} + (\cos^2 \theta_I \cos^2 \Lambda_s \cos \Lambda_a \sin \Lambda_a) \right. \\ \left. \{ m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k) \} \right] \{ \Phi_q \} \{ \Phi_q \}^T dx$$

$$[K_{41}] = \int_0^{l_e} \left[ m (\cos^2 \theta_I \sin \Lambda_s \cos \Lambda_s \sin^2 \Lambda_s) \{ \Phi_q \} \{ \Phi_c \}^T \right. \\ \left. + (\cos^2 \theta_I \sin^2 \Lambda_s \cos \Lambda_a - \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a) \right. \\ \left. \{ m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k) \} \{ \Phi_q \} \{ \Phi'_c \}^T \right] dx$$

$$[K_{42}] = \int_0^{l_e} \left[ m (\cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a \sin \Lambda_a) \{ \Phi_q \} \{ \Phi_c \}^T + (\cos^2 \theta_I \sin^2 \Lambda_s \cos \Lambda_a \right. \\ \left. - \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a) \{ m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k) \} \{ \Phi_q \} \{ \Phi'_c \}^T \right] dx$$

$$[K_{43}] = 0$$

$$[K_{44}] = \int_0^{l_e} \left[ -m (\cos^2 \theta_I \cos^2 \Lambda_a + \cos^2 \theta_I \sin^2 \Lambda_s \sin^2 \Lambda_a) \{ \Phi_q \} \{ \Phi_q \}^T \right] dx$$

#### 6.2.4 Vector $[V^L]_{14_1}$ :

$$\begin{aligned}
 [V_{11}^L] = & \int_0^{l_e} \left[ -\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\} \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a \{\Phi_c\} \right. \\
 & \left. - \{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\} \cos \Lambda_s \{\Phi_c\} + \{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\} \right. \\
 & \left. \cos^2 \Lambda_s \cos \Lambda_a \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \{\Phi'_c\} \right] dx
 \end{aligned}$$

$$[V_{21}^L] = \int_0^{l_e} \left[ \left\{ -m \left( e_1 + e_2 + \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \cos^2 \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) \{\Phi_c\} \right. \right.$$

$$\left. \left. + m\zeta_m \cos(\theta_G + \phi_k) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \cos \theta_I \cos \Lambda_s \{\Phi'_c\} \right\} \right] dx$$

$$\begin{aligned}
 [V_{31}^L] = & \int_0^{l_e} \left[ \left\{ (I_{\zeta\zeta} - I_{\eta\eta}) \sin(\theta_G + \phi_k) \cos(\theta_G + \phi_k) \cos^2 \theta_I \cos^2 \Lambda_s \cos^2 \Lambda_a \right. \right. \\
 & - I_{\eta\zeta} \left\{ \sin^2(\theta_G + \phi_k) - \cos^2(\theta_G + \phi_k) \right\} \cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a + \{I_{\zeta\zeta} \sin(\theta_G + \phi_k) \right. \\
 & \left. \left. + I_{\eta\eta} \cos(\theta_G + \phi_k)\} \sin \theta_I \cos \Lambda_a \right\} \{\Phi_q\} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 [V_{41}^L] = & \int_0^{l_e} \left[ -m \left\{ e_1 + e_2 + \sum_{i=1}^{n-1} (l_e)_i + x_k + (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \right. \right. \\
 & \left. \left. - \alpha (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right\} \cos \theta_I \cos \Lambda_s \cos \Lambda_a \{\Phi_q\} \right] dx
 \end{aligned}$$

Using small angle approximation for  $\phi_k$ , the vector  $[V_{31}^L]$  can be written in a modified form, which is given in Appendix A.

### 6.2.5 Mass Matrix $[M^1]_{14,3}$ :

$$\begin{aligned}
[M^1_{11}] &= \int_0^{l_e} [\cos \Lambda_s \{ \sin \psi_k \cos \theta_I - \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) \} - \\
&\quad \sin \Lambda_s \cos \Lambda_a \{ \cos \psi_k + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \} - \sin \Lambda_s \sin \Lambda_a \\
&\quad \{ \sin \psi_k \sin \theta_I + \cos \psi_k (\beta_s \sin \theta_I - \beta_d \cos \theta_I - \beta_p \cos \theta_I) \}] \{ \Phi_c \} dx \\
[M^1_{12}] &= \int_0^{l_e} [\sin \Lambda_s \cos \Lambda_a \{ \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) - \sin \psi_k \} - \cos \Lambda_s \{ \cos \psi_k \cos \theta_I \\
&\quad + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) \} + \sin \Lambda_s \sin \Lambda_a \{ \cos \psi_k \sin \theta_I - \\
&\quad \{ \sin \psi_k (\beta_s \sin \theta_I - \beta_d \cos \theta_I - \beta_p \cos \theta_I) \}] \{ \Phi_c \} dx \\
[M^1_{13}] &= \int_0^{l_e} -[\cos \Lambda_a \sin \Lambda_s \{ \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \} + \sin \theta_I \cos \Lambda_s + \cos \theta_I \sin \Lambda_s \\
&\quad \sin \Lambda_a] \{ \Phi_c \} dx \\
[M^1_{21}] &= \int_0^{l_e} [\sin \Lambda_a \{ \cos \psi_k + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \} + \cos \Lambda_a \{ \cos \psi_k (\beta_d \cos \theta_I - \\
&\quad \beta_s \sin \theta_I + \beta_p \cos \theta_I) - \sin \psi_k \sin \theta_I \}] \{ \Phi_c \} dx \\
[M^1_{22}] &= \int_0^{l_e} [\sin \Lambda_a \{ \sin \psi_k - \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \} + \cos \Lambda_a \{ \cos \psi_k \sin \theta_I \\
&\quad - \sin \psi_k (\beta_s \sin \theta_I - \beta_d \cos \theta_I - \beta_p \cos \theta_I) \}] \{ \Phi_c \} dx \\
[M^1_{23}] &= \int_0^{l_e} [\sin \Lambda_a \{ \beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \} - \cos \theta_I \cos \Lambda_a] \{ \Phi_c \} dx \\
[M^1_{31}] &= 0 \\
[M^1_{32}] &= 0 \\
[M^1_{33}] &= 0 \\
[M^1_{41}] &= \int_0^{l_e} [\sin \Lambda_s \cos \psi_k \{ \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) - \beta_p \sin \theta_I \} - \sin \psi_k \cos \theta_I \} - \\
&\quad \cos \Lambda_s \cos \Lambda_a \{ \cos \psi_k + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \} + \cos \Lambda_s \sin \Lambda_a \\
&\quad \{ \cos \psi_k (\beta_d \cos \theta_I - \beta_s \sin \theta_I + \beta_p \cos \theta_I) - \sin \psi_k \sin \theta_I \}] \{ \Phi_q \} dx
\end{aligned}$$

$$\begin{aligned}
\left[ M_{42}^1 \right] &= \int_0^{l_e} \left[ \sin \Lambda_s \left\{ \cos \psi_k \cos \theta_I + \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I - \beta_p \sin \theta_I) - \cos \Lambda_s \cos \Lambda_a \right. \right. \\
&\quad \left. \left. \left\{ \sin \psi_k - \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right\} + \cos \Lambda_s \sin \Lambda_a \left\{ \cos \psi_k \sin \theta_I \right. \right. \right. \\
&\quad \left. \left. \left. + \sin \psi_k (\beta_d \cos \theta_I - \beta_s \sin \theta_I + \beta_p \cos \theta_I) \right\} \right] \{\Phi_q\} dx \\
\left[ M_{43}^1 \right] &= \int_0^{l_e} \left[ \sin \theta_I \sin \Lambda_s - \cos \Lambda_s \cos \Lambda_a \left\{ \sin \psi_k - \cos \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right\} - \right. \\
&\quad \left. \cos \theta_I \cos \Lambda_a \sin \Lambda_a \right] \{\Phi_q\} dx
\end{aligned}$$

### 6.2.6 Mass Matrix $\left[ M^2 \right]_{14-3}$ :

$$\begin{aligned}
\left[ M_{11}^2 \right] &= \int_0^{l_e} \left[ \dot{\theta}_I \cos \psi_k \left\{ \beta_s (\cos \theta_I \sin \Lambda_s \sin \Lambda_a - \sin \theta_I \cos \Lambda_s) + \beta_d (\sin \theta_I \sin \Lambda_s \sin \Lambda_a \right. \right. \\
&\quad \left. \left. + \cos \theta_I \cos \Lambda_s) \right\} - 2\dot{\theta}_I \sin \Lambda_s \sin \Lambda_a \left\{ \sin \psi_k (\beta_d \sin \theta_I + \beta_s \cos \theta_I) - \right. \right. \\
&\quad \left. \left. \sin \psi_k \sin \theta_I \sin \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I) + (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \right\} \right. \\
&\quad \left. \dot{\theta}_I \sin \psi_k \cos \theta_I \cos \Lambda_a \sin \Lambda_s - 2\dot{\theta}_I \cos \Lambda_s \cos \psi_k - \dot{\theta}_I \cos \Lambda_a \sin \Lambda_s \right] \{\Phi_c\} dx \\
\left[ M_{12}^2 \right] &= \int_0^{l_e} \left[ \dot{\theta}_I \sin \psi_k \left\{ \beta_s (\cos \theta_I \sin \Lambda_s \sin \Lambda_a - \sin \theta_I \cos \Lambda_s) + \beta_d (\sin \theta_I \sin \Lambda_s \sin \Lambda_a \right. \right. \\
&\quad \left. \left. + \cos \theta_I \cos \Lambda_s) \right\} - 2\dot{\theta}_I \sin \Lambda_s \sin \Lambda_a \left\{ \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) + \dot{\theta}_I \sin \Lambda_a \cos \Lambda_a \right. \right. \\
&\quad \left. \left. (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \cos \psi_k \sin \theta_I + (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \right\} \right. \\
&\quad \left. - 2\dot{\theta}_I \cos \Lambda_s \sin \psi_k + \dot{\theta}_I \cos \psi_k \cos \Lambda_a \sin \Lambda_s \right] \{\Phi_c\} dx \\
\left[ M_{13}^2 \right] &= \int_0^{l_e} \left[ -\dot{\theta}_I \sin \theta_I \left\{ \sin \Lambda_a \sin \Lambda_s + \cos \Lambda_a \sin \Lambda_s (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \right\} + \right. \\
&\quad \left. (\dot{\theta}_I) \cos \theta_I [\cos \Lambda_s - \cos \Lambda_a \sin \Lambda_s (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] + (\dot{\theta}_I) \cos \Lambda_a \sin \Lambda_s \right. \\
&\quad \left. (\beta_d \sin \theta_I + \beta_s \cos \theta_I) + \dot{\theta}_I \sin \theta_I \sin \Lambda_s \sin \Lambda_a - \dot{\theta}_I \cos \theta_I \cos \Lambda_a \right] \{\Phi_c\} dx \\
\left[ M_{21}^2 \right] &= \int_0^{l_e} \left[ (\dot{\theta}_I) \cos \psi_k (\beta_s \cos \theta_I \cos \Lambda_a + \beta_d \sin \theta_I \cos \Lambda_a) - 2(\dot{\theta}_I) [\cos \psi_k (\beta_s \cos \theta_I + \right. \right. \\
&\quad \left. \left. \beta_d \sin \theta_I] \cos \Lambda_a + (\sin \psi_k \cos \theta_I) \dot{\theta}_I \sin \Lambda_a (\beta_s \sin \theta_I - \beta_d \cos \theta_I) + \dot{\theta}_I \sin \Lambda_a \right. \right. \\
&\quad \left. \left. \sin \theta_I [\sin \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] + (\dot{\theta}_I) \sin \psi_k (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \right\} \right] \{\Phi_c\} dx
\end{aligned}$$

$$\begin{aligned} \left[ M_{22}^2 \right] &= \int_0^{(l_e)} \left[ \dot{\theta}_I \sin \psi_k (\beta_s \cos \theta_I \cos \Lambda_a + \beta_d \sin \theta_I \cos \Lambda_a) - 2(\dot{\theta}_I) \cos \Lambda_a \sin \psi_k (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \right. \\ &\quad \left. - \dot{\theta}_I \cos \psi_k \sin \Lambda_a [\beta_s \sin 2\theta_I - \beta_d \cos 2\theta_I] + \right. \\ &\quad \left. \dot{\theta}_I \sin \Lambda_a \cos \psi_k (\beta_s \sin \theta_I - \beta_d \cos 2\theta_I) \right] dx \end{aligned}$$

$$\begin{aligned} \left[ M_{23}^2 \right] &= \int_0^{(l_e)} - \left[ \dot{\theta}_I \sin \theta_I \cos \Lambda_a + \dot{\theta}_I \sin \Lambda_a (\beta_s) + \dot{\theta}_I \sin \Lambda_a (\beta_d \sin \theta_I + \beta_s \cos \theta_I) \right. \\ &\quad \left. \dot{\theta}_I \sin \theta_I \cos \Lambda_a \{\Phi_c\} \right] dx \end{aligned}$$

$$\left[ M_{32}^2 \right] = 0$$

$$\left[ M_{31}^2 \right] = 0$$

$$\left[ M_{33}^2 \right] = 0$$

$$\begin{aligned} \left[ M_{41}^2 \right] &= \int_0^{(l_e)} \left[ \dot{\theta}_I \cos \psi_k [\beta_s (\cos \theta_I \cos \Lambda_s \sin \Lambda_a + \sin \theta_I \sin \Lambda_s) + \beta_d (\sin \theta_I \cos \Lambda_s \sin \Lambda_a \right. \\ &\quad \left. - \cos \theta_I \cos \Lambda_s)] - 2\dot{\theta}_I \cos \Lambda_s \sin \Lambda_a \cos \psi_k (\beta_d \sin \theta_I + \beta_s \cos \theta_I) + \right. \\ &\quad \left. \dot{\theta}_I \sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I) - 2\dot{\theta}_I \cos \psi_k \sin \Lambda_s (\beta_s \sin \theta_I \right. \\ &\quad \left. - \beta_d \cos \theta_I) - \dot{\theta}_I (\sin \psi_k \sin \theta_I) \cos \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I) + \right. \\ &\quad \left. \cos \Lambda_s \cos \Lambda_a \dot{\theta}_I \sin \psi_k (\beta_s \sin \theta_I - \beta_d \cos \theta_I) \right] \{\Phi_q\} dx \end{aligned}$$

$$\begin{aligned} \left[ M_{42}^2 \right] &= \int_0^{(l_e)} \left[ \dot{\theta}_I \sin \psi_k [\beta_s (\cos \theta_I \cos \Lambda_s \sin \Lambda_a + \sin \theta_I \sin \Lambda_s) + \beta_d (\sin \theta_I \cos \Lambda_s \sin \Lambda_a \right. \\ &\quad \left. - \cos \theta_I \sin \Lambda_s)] + \dot{\theta}_I \cos \Lambda_s \cos \Lambda_a \cos \psi_k \cos \theta_I (\beta_s \sin \theta_I - \beta_d \cos \theta_I) - \right. \\ &\quad \left. 2\dot{\theta}_I \cos \Lambda_s \sin \psi_k \sin \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I) + \dot{\theta}_I \cos \psi_k \sin \theta_I \cos \Lambda_s \cos \Lambda_a \right. \\ &\quad \left. (\beta_s \cos \theta_I + \beta_d \sin \theta_I) - 2\dot{\theta}_I \sin \Lambda_s \sin \psi_k (\beta_s \sin \theta_I - \beta_d \cos \theta_I) + \dot{\theta}_I \cos \psi_k \cos \theta_I \right. \\ &\quad \left. \cos \Lambda_s \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I) \right] \} \end{aligned}$$

$$\begin{aligned} \left[ M_{43}^2 \right] &= \int_0^{(l_e)} \left[ -(\dot{\theta}_I) \sin \theta_I [\cos \Lambda_s \cos \Lambda_a (\beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \sin \Lambda_a] - (\dot{\theta}_I) \cos \theta_I \right. \\ &\quad \left. [\sin \Lambda_s + \cos \Lambda_s \cos \Lambda_a (\beta_s \cos \theta_I + \beta_d \sin \theta_I)] + \cos \Lambda_s \cos \Lambda_a \dot{\theta}_I (\beta_s \cos \theta_I + \right. \\ &\quad \left. \beta_d \sin \theta_I) + \dot{\theta}_I \cos \theta_I \sin \Lambda_s + \dot{\theta}_I \sin \theta_I \cos \Lambda_s \sin \Lambda_a \right] \{\Phi_q\} dx \end{aligned}$$

### 6.2.7 Mass Matrix $\left[M^3\right]_{14 \times 3}$ :

$$\left[M_{11}^3\right] = \int_0^{l_e} \left\{ (x_k) \left[ \sin \psi_k \sin \Lambda_s \sin \Lambda_a (\sin \theta_I \sin \Lambda_s - \cos \theta_I \cos \Lambda_s \cos \Lambda_a) - \right. \right. \\ \left. \left. \cos \Lambda_s (\cos \Lambda_s \cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_s \cos \Lambda_a) + \sin \Lambda_s \sin \Lambda_a \right. \right. \\ \left. \left. (\sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a - \cos \psi_k \sin \Lambda_s) \right] + (a) [\sin \theta_I \cos \psi_k \cos \Lambda_s + \right. \\ \left. \cos \theta_I \cos \psi_k \sin \Lambda_s \sin \Lambda_a] + (e_1 + e_2 + \sum (u)_i) [\sin \psi_k \sin \Lambda_I \cos \Lambda_s + \right. \\ \left. \sin \psi_k \sin \Lambda_s \sin \Lambda_a \cos \theta_I] \right\} \{\Phi_c\} dx$$

$$\left[M_{12}^3\right] = \int_0^{l_e} \left\{ (x_k) \left[ \cos \psi_k \cos \Lambda_a \sin \Lambda_s (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s) - \right. \right. \\ \left. \left. \cos^2 \Lambda_s (\cos \Lambda_a \cos \psi_k \sin \theta_I + \sin \Lambda_a \sin \psi_k) - \sin \Lambda_s \sin \Lambda_a \right. \right. \\ \left. \left. (\sin \psi_k \sin \Lambda_s + \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) \right] + (a) [\sin \theta_I \sin \psi_k \cos \Lambda_s + \right. \\ \left. \cos \theta_I \sin \psi_k \sin \Lambda_s \sin \Lambda_a] - (e_1 + e_2 + \sum (u)_i) [\cos \psi_k \sin \theta_I \cos \Lambda_s + \right. \\ \left. \cos \psi_k \cos \theta_I \sin \Lambda_s \sin \Lambda_a] \right\} \{\Phi_c\} dx$$

$$\left[M_{21}^3\right] = \int_0^{l_e} \left\{ (x_k) \left[ \sin \psi_k \sin \Lambda_a (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s) - \right. \right. \\ \left. \left. \cos \Lambda_a (\sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a - \cos \psi_k \sin \Lambda_s) \right] + (a) \cos \theta_I \cos \psi_k \cos \Lambda_a \right. \\ \left. + (e_1 + e_2 + \sum (u)_i) \sin \psi_k \cos \theta_I \cos \Lambda_a \right\} \{\Phi_c\} dx$$

$$\left[M_{22}^3\right] = \int_0^{l_e} \left\{ (x_k) \left[ \cos \psi_k \sin \Lambda_a (\sin \theta_I \sin \Lambda_s - \cos \theta_I \cos \Lambda_s \sin \Lambda_a) - \right. \right. \\ \left. \left. \cos \Lambda_a (\sin \psi_k \sin \Lambda_s - \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) \right] + (a) \cos \theta_I \cos \Lambda_a \sin \psi_k \right. \\ \left. - (e_1 + e_2 + \sum (u)_i) \cos \psi_k \cos \theta_I \cos \Lambda_a \right\} \{\Phi_c\} dx$$

$$\left[M_{23}^3\right] = \int_0^{l_e} \left\{ -(x_k) \left[ \sin \Lambda_a (\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \cos \theta_I \sin \Lambda_s) + \cos \Lambda_a [\sin \Lambda_s (\beta_p + \right. \right. \\ \left. \left. + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I] \right] + (a) \sin \Lambda_a - \right. \\ \left. (e_1 + e_2 + \sum (u)_i) \sin \theta_I \cos \Lambda_a \right\} \{\Phi_c\} dx$$

$$\left[M_{31}^3\right] = 0$$

$$\left[M_{32}^3\right] = 0$$

$$\left[M_{33}^3\right] = 0$$

$$\begin{aligned}
\left[ M_{41}^3 \right] = \int_0^L \{ & (x_k) \left[ \sin \psi_k \cos \Lambda_s \cos \Lambda_a (\sin \theta_I \sin \Lambda_s - \cos \theta_I \cos \Lambda_s \sin \Lambda_a) + \right. \\
& \cos \Lambda_s \sin \Lambda_s (\cos \psi_k \sin \Lambda_a - \sin \theta_I \sin \psi_k \cos \Lambda_a) + \cos \Lambda_s \sin \Lambda_a \\
& (\sin \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a - \cos \psi_k \sin \Lambda_s) \left. \right] - (a) \left[ \sin \theta_I \cos \psi_k \sin \Lambda_s - \right. \\
& \cos \theta_I \cos \psi_k \sin \Lambda_a \cos \Lambda_s \left. \right] - (e_1 + e_2 + \sum (u)_i) \left[ \sin \psi_k \sin \theta_I \sin \Lambda_s - \right. \\
& \sin \psi_k \cos \Lambda_s \sin \Lambda_a \cos \theta_I \left. \right] \} \{ \Phi_q \} dx
\end{aligned}$$

$$\begin{aligned}
\left[ M_{42}^3 \right] = \int_0^L \{ & (x_k) \left[ \cos \psi_k \cos \Lambda_a \cos \Lambda_s (\cos \theta_I \cos \Lambda_s \sin \Lambda_a - \sin \theta_I \sin \Lambda_s) + \right. \\
& \cos \Lambda_s \sin \Lambda_s (\sin \psi_k \sin \Lambda_a + \sin \theta_I \cos \psi_k \cos \Lambda_a) - \cos \Lambda_s \sin \Lambda_a \\
& (\sin \psi_k \sin \Lambda_s + \cos \psi_k \cos \theta_I \cos \Lambda_s \cos \Lambda_a) \left. \right] - (a) \left[ \sin \theta_I \sin \psi_k \sin \Lambda_s - \right. \\
& \cos \theta_I \sin \psi_k \sin \Lambda_a \cos \Lambda_s \left. \right] + (e_1 + e_2 + \sum (u)_i) \left[ \cos \psi_k \sin \theta_I \sin \Lambda_s - \right. \\
& \cos \psi_k \cos \Lambda_s \sin \Lambda_a \cos \theta_I \left. \right] \} \{ \Phi_q \} dx
\end{aligned}$$

$$\begin{aligned}
\left[ M_{43}^3 \right] = \int_0^L \{ & (x_k) \left[ \cos \Lambda_s \cos \Lambda_a (\sin \theta_I \cos \Lambda_s \sin \Lambda_a + \cos \theta_I \sin \Lambda_s) + \right. \\
& \cos \Lambda_s \sin \Lambda_s \left[ \sin \Lambda_a (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) - \cos \theta_I \cos \Lambda_a \right] - \\
& \cos \Lambda_s \sin \Lambda_a \left[ \sin \Lambda_s (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I) + \cos \Lambda_s \cos \Lambda_a \sin \theta_I \right] \left. \right] - (a) \left[ \cos \Lambda_s \cos \Lambda_a \right] - \\
& (e_1 + e_2 + \sum (u)_i) (\cos \theta_I \sin \Lambda_a + \cos \Lambda_s \sin \Lambda_a \sin \theta_I) \} \{ \Phi_q \} dx
\end{aligned}$$

### 6.2.8 Matrix $[\mathbf{M}^4]_{14 \times 3}$ :

$$\begin{aligned}
\left[ M_{11}^4 \right] = \int_0^L \{ & [-m \cos \Lambda_s \cos \Lambda_a \{ a \cos \theta_I - (\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \right. \\
& \left. - (e_1 + e_2) \cos \psi_k + m \cos \Lambda_s \cos \Lambda_a \left\{ \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \sin \psi_k \right\} \right] \} \{ \Phi_c \} dx
\end{aligned}$$

$$\begin{aligned}
\left[ M_{12}^4 \right] = \int_0^L \{ & [-m \cos \Lambda_s \cos \Lambda_a a \cos \theta_I - (\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \right. \\
& \left. - (e_1 + e_2) \sin \psi_k - m \cos \Lambda_s \cos \Lambda_a \left\{ \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \cos \psi_k \right\} \right] \} \{ \Phi_c \} dx
\end{aligned}$$

$$\begin{aligned}
\left[ M_{13}^4 \right] &= \int_0^{l_e} [ma \cos \Lambda_s \cos \theta_I (2 \cos \theta_I - \sin \theta_I) \{\Phi_c\} dx \\
\left[ M_{21}^4 \right] &= \int_0^{l_e} [-m \cos \Lambda_s \cos \Lambda_a \cos \theta_I \{2a \sin \psi_k \cos \theta_I + \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \cos \psi_k\} \{\Phi_c\} dx \\
\left[ M_{22}^4 \right] &= \int_0^{l_e} [m \sin \Lambda_a \cos \Lambda_a \cos \Lambda_s \{2a \sin \psi_k \cos \theta_I - \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \sin \psi_k\} \{\Phi_c\} dx \\
\left[ M_{23}^4 \right] &= \int_0^{l_e} [-m \cos \Lambda_a \cos^2 \Lambda_s \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) (\beta_p + \beta_d \cos \theta_I - \beta_s \sin \theta_I \dot{\theta}_I) \{\Phi_c\} dx \\
\left[ M_{32}^4 \right] &= 0 \\
\left[ M_{33}^4 \right] &= 0 \\
\left[ M_{31}^4 \right] &= 0 \\
\left[ M_{41}^4 \right] &= \int_0^{l_e} [-m \cos \Lambda_s \cos \Lambda_a \cos \theta_I \{a \cos \theta_I - (\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \\
&\quad - (e_1 + e_2) \sin \psi_k \{\Phi_q\}\} dx \\
\left[ M_{42}^4 \right] &= \int_0^{l_e} [m \cos \Lambda_s \cos \theta_I \{a \cos \theta_I - (\beta_s \cos \theta_I - \beta_d \sin \theta_I + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \\
&\quad - (e_1 + e_2) \cos \psi_k \{\Phi_q\}\} dx \\
\left[ M_{43}^4 \right] &= \int_0^{l_e} m [\{(e_1 + e_2) + 2 \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right)\} \cos \theta_I \cos \Lambda_s \cos^2 \Lambda_a \{\Phi_a\}] dx
\end{aligned}$$

### 6.2.9 Vector $\left[ V^I \right]_{14 \times 1}$ :

$$\begin{aligned}
\left[ V_{11}^I \right] &= \int_0^{l_e} [-m \cos \Lambda_s \cos \Lambda_a \{3a - (e_1 + e_2) - ((\beta_s \cos \theta_I + \beta_d \sin \theta_I) + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \\
&\quad - (\beta_d (\cos \theta_I + \sin \theta_I) + \beta_s (\sin \theta_I - \cos \theta_I))\} \cos \Lambda_s \dot{\theta}_I \\
&\quad - \{((\beta_s \cos \theta_I + \beta_d \sin \theta_I) - 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right)\} \ddot{\theta}_I \cos^2 \Lambda_s \cos \Lambda_a \\
&\quad + \{a + (\beta_s \cos \theta_I + \beta_d \sin \theta_I) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \dot{\theta}_I^2 \cos \Lambda_a \cos \Lambda_s\} \{\Phi_c\}] dx
\end{aligned}$$

$$\begin{aligned}
[V_{21}^I] = \int_0^{l_e} & [-m[\{\alpha + (\beta_s \cos \theta_I + \beta_d \sin \theta_I)(\sum_{i=1}^{n-1} (l_e)_i + x_k)\} \dot{\theta}_I \\
& - \{\alpha \cos \theta_I \cos \Lambda_s \cos \Lambda_a - ((\beta_s \cos \theta_I + \beta_d \sin \theta_I) + 1) \left( \sum_{i=1}^{n-1} (l_e)_i + x_k \right) \\
& - (e_1 + e_2)\} \ddot{\theta}_I \cos \Lambda_s \sin \Lambda_a \sin \Lambda_s \\
& - (\beta_s \cos \theta_I + \beta_d \sin \theta_I)(\sum_{i=1}^{n-1} (l_e)_i + x_k) \dot{\theta}_I^2 \cos \Lambda_s] \{\Phi_c\} dx
\end{aligned}$$

$$[V_{31}^I] = 0$$

$$\begin{aligned}
[V_{41}^I] = \int_0^{l_e} & [-m[\{2(\sum_{i=1}^{n-1} (l_e)_i + x_k)(\beta_s \cos \theta_I + \beta_d \sin \theta_I)\} \dot{\theta}_I \cos \Lambda_s \cos \Lambda_a + \alpha \ddot{\theta}_I \\
& - \{(e_1 + e_2 + \sum_{i=1}^{n-1} (l_e)_i + x_k)(1 + (\beta_s \cos \theta_I + \beta_d \sin \theta_I)) - (e_1 + e_2 - \alpha) \dot{\theta}_I^2 \cos^2 \Lambda_a \\
& + \alpha \sin \theta_I \cos \theta_I \dot{\theta}_I \cos \Lambda_a \cos \Lambda_s\} \{\Phi_q\} dx
\end{aligned}$$

The integrals of the shape functions, which are needed for the evaluation of the mass matrix, are given below:

$$\int_0^l x \{\phi_c\} \{\phi_c\}^T dx = \left( l_e^2 \right) \begin{bmatrix} \frac{3}{35} & \frac{l_e}{60} & \frac{9}{140} & -\frac{l_e}{70} \\ \frac{l_e}{60} & \frac{l_e^2}{280} & \frac{l_e}{60} & -\frac{l_e^2}{280} \\ \frac{9}{140} & \frac{l_e}{60} & \frac{2}{7} & -\frac{l_e}{28} \\ -\frac{l_e}{70} & -\frac{l_e^2}{280} & -\frac{l_e}{28} & \frac{l_e^2}{168} \end{bmatrix}$$

$$\int_0^l x \{\phi_c\} \{\phi_q\}^T dx = \left( \frac{l_e^2}{420} \right) \begin{bmatrix} 9 & 52 & 2 \\ l_e & 12l_e & l_e \\ -9 & 88 & 68 \\ l_e & -16l_e & -6l_e \end{bmatrix}$$

$$\int_0^l x \{\phi'_c\} \{\phi_q\}^T dx = \left( \frac{l_e}{60} \right) \begin{bmatrix} 0 & -24 & -6 \\ l_e & -4l_e & -2l_e \\ 0 & 24 & 6 \\ -l_e & 0 & 6l_e \end{bmatrix}$$

$$\int_0^l x \{\phi_q\} \{\phi_q\}^T dx = \left( \frac{l_e^2}{60} \right) \begin{bmatrix} 1 & 0 & -1 \\ 0 & 16 & 4 \\ -1 & 4 & 7 \end{bmatrix}$$

$$\int_0^l \{\Phi_c\} \{\Phi_c\}^T dx = \left( \frac{l_e}{420} \right) \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi'_c\} \{\Phi_c\}^T dx = \left( \frac{1}{60} \right) \begin{bmatrix} -30 & -6l_e & -30 & 6l_e \\ 6l_e & 0 & -6l_e & l_e^2 \\ 30 & 6l_e & 30 & -6l_e \\ -6l_e & -l_e^2 & 6l_e & 0 \end{bmatrix}$$

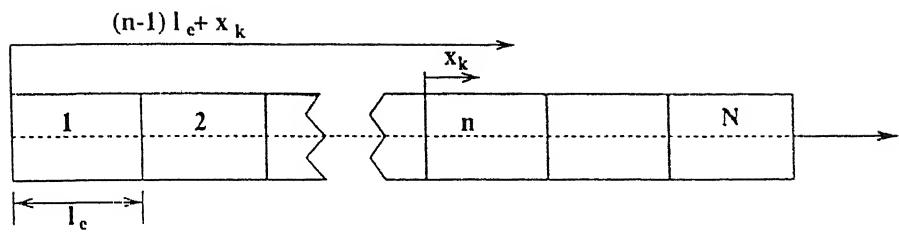
$$\int_0^{l_e} \{\Phi_q\} \{\Phi_q\}^T dx = \left( \frac{l_e}{30} \right) \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c\} \{\Phi_q\}^T dx = \left( \frac{l_e}{60} \right) \begin{bmatrix} 11 & 20 & -1 \\ l_e & 4l_e & 0 \\ -1 & 20 & 11 \\ 0 & -4l_e & -l_e \end{bmatrix}$$

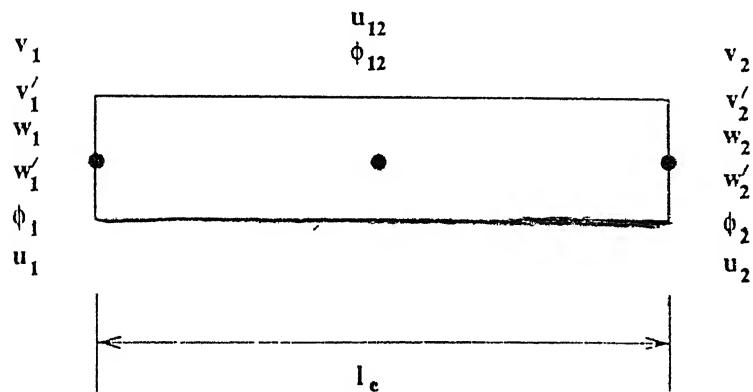
$$\int_0^{l_e} \{\Phi'_c\} \{\Phi_q\}^T dx = \left( \frac{1}{60} \right) \begin{bmatrix} -6 & -48 & -6 \\ 7l_e & -4l_e & -3l_e \\ 6 & 48 & 6 \\ -3l_e & -4l_e & 7l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c\} dx = \left( \frac{l_e}{12} \right) \begin{bmatrix} 6 \\ l_e \\ 6 \\ -l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_q\} dx = \left( \frac{l_e}{6} \right) \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \quad . \quad (6.8)$$



**Figure 6.1** Finite element model of a blade



**Figure 6.2** Element nodal degrees of freedom

## Chapter 7

### Formulation of Element Matrices Associated with Strain Energy Variation

The elemental matrix associated with the strain energy variation is derived by substituting the assumed expressions for the displacement function in the strain energy variation  $\delta U_i$ , and carrying out the integration over the length of the element. The resulting variation of the strain energy can be written in the form:

$$\delta U_i = \{\delta q\}^T \left\{ \left[ K^E \right] \{q\} + \{F^E\} \right\} \quad (7.1)$$

Where,  $\left[ K^E \right]_{14 \times 14}$  is the elemental stiffness matrix,  $\{F^E\}_{14}$  is the nonlinear stiffness vector and  $\{q\}$  represents the vector of unknown degree of freedom and is of the form:

$$\{q\} = \{v_1 v_1' v_2 v_2' w_1 w_1' w_2 w_2' \phi_1 \phi_2 \phi_3 u_1 u_2 u_3\}^T$$

The nodal degrees of freedom are shown in Fig. 6.2

#### 7.1 Linear Stiffness Matrix

$$\begin{aligned} \left[ K_{11}^E \right] &= \int_0^L \left[ \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \{\Phi_c''\} \{\Phi_c''\}^T \right] dx \\ \left[ K_{12}^E \right] &= \left[ K_{21}^E \right] = \int_0^L \left[ \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \{\Phi_c''\} \{\Phi_c''\}^T \right] dx \\ \left[ K_{13}^E \right] &= \left[ K_{31}^E \right] = \int_0^L \left[ \overline{EID_i} \{\Phi_c''\} \{\Phi_q''\}^T + (\tau_0 \overline{EAD'_i}) \{\Phi_c''\} \{\Phi_q'\}^T \right] dx \\ \left[ K_{14}^E \right] &= \left[ K_{41}^E \right] = \int_0^L \left[ \left( -\overline{EI\eta_a} \right) \{\Phi_c''\} \{\Phi_q'\}^T \right] dx \\ \left[ K_{22}^E \right] &= \int_0^L \left[ \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \{\Phi_c''\} \{\Phi_c''\}^T \right] dx \end{aligned}$$

$$\begin{aligned}
\left[ K_{23}^E \right] &= \left[ K_{32}^E \right] = \int_0^l \left[ \overline{EAD}_2 \left\{ \Phi_c'' \right\} \left\{ \Phi_q'' \right\}^T + \left( \tau_0 \overline{EAD}'_2 \right) \left\{ \Phi_c'' \right\} \left\{ \Phi_q' \right\}^T \right] dx \\
\left[ K_{24}^E \right] &= \left[ K_{42}^E \right] = \int_0^l \left[ \left( -\overline{EA\zeta}_a \right) \left\{ \Phi_c'' \right\} \left\{ \Phi_q' \right\}^T \right] dx \\
\left[ K_{33}^E \right] &= \int_0^l \left[ \left( EAD_3 \right) \left\{ \Phi_q'' \right\} \left\{ \Phi_q'' \right\}^T + \tau_0 EAD_5 \left\{ \Phi_q'' \right\} \left\{ \Phi_q' \right\}^T + \tau_0 EAD_5 \left\{ \Phi_q'' \right\} \left\{ \Phi_q' \right\}^T + \right. \\
&\quad \left. \left( \tau_0^2 EAD'_3 - GJ \right) \left\{ \Phi_q' \right\} \left\{ \Phi_q' \right\}^T \right] dx \\
\left[ K_{34}^E \right] &= \left[ K_{43}^E \right]^T = \int_0^l \left[ \left( -\overline{EAD}_0 \right) \left\{ \Phi_q'' \right\} \left\{ \Phi_q' \right\}^T + \left( -\tau_0 \overline{EAD}'_0 \right) \left\{ \Phi_q' \right\} \left\{ \Phi_q' \right\}^T \right] dx \\
\left[ K_{44}^E \right] &= \int_0^l \left[ \left( EA \right) \left\{ \Phi_q' \right\} \left\{ \Phi_q' \right\}^T \right] dx
\end{aligned} \tag{7.2}$$

If the properties like  $EI$ ,  $EI_{\eta\eta}$ ,  $EI_{\zeta\zeta}$  etc. vary linearly along the element length; various quantities defined in the linear stiffness matrices can be expressed as:

$$\begin{aligned}
\overline{EI}_{\zeta\zeta} &= \left[ EI_{\zeta\zeta_1} + \left( \frac{EI_{\zeta\zeta_1} - EI_{\zeta\zeta_2}}{l} \right) x \right] \cos \theta_G - \left[ EI_{\zeta\eta} + \left( \frac{EI_{\zeta\eta_1} - EI_{\zeta\eta_2}}{l} \right) x \right] \sin \theta_G \\
\overline{EI}_{\eta\zeta} &= \left[ EI_{\eta\zeta_1} + \left( \frac{EI_{\eta\zeta_1} - EI_{\eta\zeta_2}}{l} \right) x \right] \cos \theta_G - \left[ EI_{\eta\eta} + \left( \frac{EI_{\eta\eta_1} - EI_{\eta\eta_2}}{l} \right) x \right] \sin \theta_G \\
\overline{EI}_{\zeta\eta} &= \left[ EI_{\zeta\zeta_1} + \left( \frac{EI_{\zeta\zeta_1} - EI_{\zeta\zeta_2}}{l} \right) x \right] \sin \theta_G + \left[ EI_{\zeta\eta} + \left( \frac{EI_{\zeta\eta_1} - EI_{\zeta\eta_2}}{l} \right) x \right] \cos \theta_G \\
\overline{EI}_{\eta\eta} &= \left[ EI_{\eta\zeta_1} + \left( \frac{EI_{\eta\zeta_1} - EI_{\eta\zeta_2}}{l} \right) x \right] \sin \theta_G + \left[ EI_{\eta\eta_1} + \left( \frac{EI_{\eta\eta_1} - EI_{\eta\eta_2}}{l} \right) x \right] \cos \theta_G \\
\overline{EAD}_1 &= \left[ EAD_{11} + \left( \frac{EAD_{12} - EAD_{11}}{l} \right) x \right] \cos \theta_G - \left[ EAD_{21} + \left( \frac{EAD_{22} - EAD_{21}}{l} \right) x \right] \sin \theta_G \\
\overline{EAD}'_1 &= \left[ EAD'_{11} + \left( \frac{EAD'_{12} - EAD'_{11}}{l} \right) x \right] \cos \theta_G - \left[ EAD'_{21} + \left( \frac{EAD'_{22} - EAD'_{21}}{l} \right) x \right] \sin \theta_G \\
\overline{EA\eta}_a &= \left[ EA\eta_{a1} + \left( \frac{EA\eta_{a2} - EA\eta_{a1}}{l} \right) x \right] \cos \theta_G - \left[ EA\eta_{a1} + \left( \frac{EA\eta_{a2} - EA\eta_{a1}}{l} \right) x \right] \sin \theta_G
\end{aligned}$$

$$\begin{aligned}
EAD_3 &= \left[ EAD_{31} + \left( \frac{EAD_{32} - EAD_{31}}{l} \right) x \right] \\
EAD_5 &= \left[ EAD_{51} + \left( \frac{EAD_{52} - EAD_{51}}{l} \right) x \right] \\
EAD'_3 &= \left[ EAD'_{31} + \left( \frac{EAD'_{32} - EAD'_{31}}{l} \right) x \right] \\
EAD_0 &= \left[ EAD_{01} + \left( \frac{EAD_{02} - EAD_{01}}{l} \right) x \right] \\
EA &= \left[ EA_1 + \left( \frac{EA_2 - EA_1}{l} \right) x \right] \\
GJ &= \left[ GJ_{01} + \left( \frac{GJ_{02} - GJ_{01}}{l} \right) x \right] + 2 \left[ GJ_{11} + \left( \frac{GJ_{12} - GJ_{11}}{l} \right) x \right] + \left[ GJ_{21} + \left( \frac{GJ_{22} - GJ_{21}}{l} \right) x \right]
\end{aligned} \tag{7.3}$$

Where, the subscript 1 and 2 refers to the properties at the two end nodes.

The integrals of the shape functions, which are needed for the evaluation of the stiffness matrix, are given below:

$$\int_0^l \{\Phi'_c\} \{\Phi'_c\}^T dx = \left( \frac{1}{30l_e} \right) \begin{bmatrix} 36 & 3l_e & -36 & 3l_e \\ 3l_e & 4l_e^2 & -3l_e & -l_e^2 \\ -36 & -3l_e & 36 & -3l_e \\ 3l_e & -l_e^2 & -3l_e & 4l_e^2 \end{bmatrix}$$

$$\int_0^l \{\Phi'_c\} \{\Phi''_c\}^T dx = \left( \frac{1}{2l_e} \right) \begin{bmatrix} 0 & 2 & 0 & -2 \\ -2 & -1 & 2 & -1 \\ 0 & -2 & 0 & 2 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c''\} \{\Phi_c''\} dx = \left( \frac{1}{l_e^3} \right) \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c'\} \{\Phi_c'\} dx = \left( \frac{1}{60} \right) \begin{bmatrix} 36 & 6l_e & -36 & 0 \\ 6l_e & 2l_e^2 & -6l_e & -l_e^2 \\ -36 & -6l_e & 36 & 0 \\ 0 & -l_e^2 & 0 & 6l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c'\} \{\Phi_c''\} dx = \left( \frac{1}{30l_e} \right) \begin{bmatrix} -18 & 6l_e & 18 & -24l_e \\ 9l_e & -2l_e^2 & 9l_e & -7l_e^2 \\ 18 & -6l_e & -18 & 24l_e \\ 21l_e & 8l_e^2 & -21l_e & 13l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c''\} \{\Phi_c''\}^T dx = \left( \frac{1}{l_e^2} \right) \begin{bmatrix} 6 & 2l_e & -6 & l_e \\ 2l_e & l_e^2 & -2l_e & l_e^2 \\ -6 & -2l_e & 6 & -4l_e \\ 4l_e & l_e^2 & -4l_e & 3l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c'\} \{\Phi_q'\}^T dx = \left( \frac{1}{3l_e} \right) \begin{bmatrix} 3 & 0 & -3 \\ -l_e & 2l_e & -l_e \\ -3 & 0 & 3 \\ l_e & -2l_e & l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c'\} \{\Phi_q''\}^T dx = \left( \frac{1}{l_e^2} \right) \begin{bmatrix} -4 & 8 & -4 \\ 0 & 0 & 0 \\ 4 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c''\} \{\Phi_q'\}^T dx = \left( \frac{1}{l_e^2} \right) \begin{bmatrix} 4 & -8 & 4 \\ 3l_e & -4l_e & l_e \\ -4 & 8 & -4 \\ l_e & -4l_e & 3l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c''\} \{\Phi_q''\}^T dx = \left( \frac{1}{l_e^3} \right) \begin{bmatrix} 0 & 0 & 0 \\ -4l_e & 8l_e & -4l_e \\ 0 & 0 & 0 \\ 4l_e & -8l_e & 4l_e \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c'\} \{\Phi_q'\} dx = \left( \frac{1}{60} \right) \begin{bmatrix} 18 & 24 & -42 \\ -l_e & 12l_e & -11l_e \\ -18 & -24 & 42 \\ 9l_e & -28l_e & 19l_e \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c'\} \{\Phi_q''\}^T dx = \left( \frac{1}{3l_e} \right) \begin{bmatrix} -6 & 12 & -6 \\ -l_e & 2l_e & -l_e \\ 6 & -12 & 6 \\ l_e & -2l_e & l_e \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c''\} \{\Phi_q'\}^T dx = \left( \frac{1}{3l_e} \right) \begin{bmatrix} 3 & -12 & 9 \\ 2l_e & -4l_e & 2l_e \\ -3 & 12 & -9 \\ l_e & -8l_e & 7l_e \end{bmatrix}$$

$$\int_0^{l_e} x \{\Phi_c''\} \{\Phi_q''\}^T dx = \left( \frac{1}{l_e^2} \right) \begin{bmatrix} 4 & -8 & 4 \\ 0 & 0 & 0 \\ -4 & 8 & -4 \\ 4l_e & -8l_e & 4l_e \end{bmatrix}$$

$$\int_0^l \{\Phi'_q\} \{\Phi'_q\} dx = \frac{1}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$\int_0^l \{\Phi'_q\} \{\Phi''_q\}^T dx = \frac{1}{l_e^2} \begin{bmatrix} -4 & 8 & -4 \\ 0 & 0 & 0 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\int_0^l \{\Phi''_q\} \{\Phi''_q\}^T dx = \frac{1}{l_e^3} \begin{bmatrix} 16 & -32 & 16 \\ -32 & 64 & -32 \\ 16 & -32 & 16 \end{bmatrix}$$

$$\int_0^l x \{\Phi'_q\} \{\Phi'_q\}^T dx = \frac{1}{6} \begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix}$$

$$\int_0^l x \{\Phi'_q\} \{\Phi''_q\}^T dx = \frac{1}{3l_e} \begin{bmatrix} -2 & 4 & -2 \\ -8 & 16 & -8 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\int_0^l x \{\Phi''_q\} \{\Phi''_q\}^T dx = \left( \frac{1}{l_e^2} \right) \begin{bmatrix} 8 & -16 & 8 \\ -16 & 32 & -16 \\ 8 & -16 & 8 \end{bmatrix}$$

$$\int_0^l x \{\Phi''_q\} \{\Phi'_q\}^T dx = \frac{1}{3l_e} \begin{bmatrix} -2 & -8 & 10 \\ 4 & 16 & -20 \\ -2 & -8 & 10 \end{bmatrix}$$

(7.4)

During integration along the length the cross-sectional constants were assumed to vary linearly along the element. Using the linear part of the eq. 5.13 ( $\delta U$  term), and replacing the variables with their corresponding nodal values multiplied with the appropriate shape functions the entries of the elemental stiffness matrix were found. The matrix entries for  $[K^E]$  were found out to be the following:

$$K_{1 \times 1} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{1 \times 2} = K_{2 \times 1} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{2}{l} \right)$$

$$K_{1 \times 3} = K_{3 \times 1} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right)$$

$$K_{1 \times 4} = K_{4 \times 1} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{4}{l} \right)$$

$$K_{1 \times 5} = K_{5 \times 1} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{1 \times 6} = K_{6 \times 1} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right)$$

$$K_{1 \times 7} = K_{7 \times 1} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right)$$

$$K_{1 \times 8} = K_{8 \times 1} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right)$$

$$K_{1 \times 9} = K_{9 \times 1} = \tau_0 \overline{EAD'_1} \left( \frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{1}{l} \right) + \overline{EAD'_1}(0) + \Delta \overline{EAD'_1} \left( \frac{4}{l^2} \right)$$

$$K_{1 \times 10} = K_{10 \times 1} = \tau_0 \overline{EAD'_1} \left( -\frac{8}{l^2} \right) + \Delta \tau_0 \overline{EAD'_1} \left( -\frac{4}{l} \right) + \overline{EAD'_1}(0) + \Delta \overline{EAD'_1} \left( -\frac{8}{l^2} \right)$$

$$K_{1 \times 11} = K_{11 \times 1} = \tau_0 \overline{EAD'_1} \left( \frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{3}{l} \right) + \overline{EAD'_1}(0) + \Delta \overline{EAD'_1} \left( \frac{4}{l^2} \right)$$

$$K_{1 \times 12} = K_{12 \times 1} = -\overline{EA\eta_a} \left( \frac{4}{l^2} \right) - \Delta \overline{EA\eta_a} \left( \frac{1}{l} \right)$$

$$K_{1 \times 13} = K_{13 \times 1} = -\overline{EA\eta_a} \left( -\frac{8}{l^2} \right) - \Delta \overline{EA\eta_a} \left( -\frac{4}{l} \right)$$

$$K_{1 \times 14} = K_{14 \times 1} = -\overline{EA\eta_a} \left( \frac{4}{l^2} \right) - \Delta \overline{EA\eta_a} \left( \frac{3}{l} \right)$$

$$K_{2 \times 2} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) (1)$$

$$K_{2 \times 3} = K_{3 \times 2} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{2}{l} \right)$$

$$K_{2 \times 4} = K_{4 \times 2} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{2}{l} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) (1)$$

$$K_{2 \times 5} = K_{5 \times 2} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right)$$

$$K_{2 \times 6} = K_{6 \times 2} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) (1)$$

$$K_{2 \times 7} = K_{7 \times 2} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{2}{l} \right)$$

$$K_{2 \times 8} = K_{8 \times 2} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) (1)$$

$$K_{2 \times 9} = K_{9 \times 2} = \tau_0 \overline{EAD'_1} \left( \frac{3}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{2}{3} \right) + \overline{EAD_1} \left( -\frac{4}{l^2} \right) + \Delta \overline{EAD_1} (0)$$

$$K_{2 \times 10} = K_{10 \times 2} = \tau_0 \overline{EAD'_1} \left( -\frac{4}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( -\frac{4}{3} \right) + \overline{EAD_1} \left( \frac{8}{l^2} \right) + \Delta \overline{EAD_1} (0)$$

$$K_{2 \times 11} = K_{11 \times 2} = \tau_0 \overline{EAD'_1} \left( \frac{1}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{2}{3} \right) + \overline{EAD_1} \left( -\frac{4}{l^2} \right) + \Delta \overline{EAD_1} (0)$$

$$K_{2 \times 12} = K_{12 \times 2} = -\overline{EA\eta_a} \left( \frac{3}{l} \right) - \Delta \overline{EA\eta_a} \left( \frac{2}{3} \right)$$

$$K_{2 \times 13} = K_{13 \times 2} = -\overline{EA\eta_a} \left( -\frac{4}{l} \right) - \Delta \overline{EA\eta_a} \left( -\frac{4}{3} \right)$$

$$K_{2 \times 14} = K_{14 \times 2} = -\overline{EA\eta_a} \left( \frac{1}{l} \right) - \Delta \overline{EA\eta_a} \left( \frac{2}{3} \right)$$

$$K_{3 \times 3} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{3 \times 4} = K_{4 \times 3} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( -\frac{4}{l} \right)$$

$$K_{3 \times 5} = K_{5 \times 3} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right)$$

$$K_{3 \times 6} = K_{6 \times 3} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{2}{l} \right)$$

$$K_{3 \times 7} = K_{7 \times 3} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{3 \times 8} = K_{8 \times 3} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{4}{l} \right)$$

$$K_{3 \times 9} = K_{9 \times 3} = \tau_0 \overline{EAD}' \left( -\frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD}' \left( -\frac{1}{l} \right) + \overline{EAD}_1(0) + \Delta \overline{EAD}_1 \left( -\frac{4}{l^2} \right)$$

$$K_{3 \times 10} = K_{10 \times 3} = \tau_0 \overline{EAD}' \left( \frac{8}{l^2} \right) + \Delta \tau_0 \overline{EAD}' \left( \frac{4}{l} \right) + \overline{EAD}_1(0) + \Delta \overline{EAD}_1 \left( \frac{8}{l^2} \right)$$

$$K_{3 \times 11} = K_{11 \times 3} = \tau_0 \overline{EAD}' \left( -\frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD}' \left( -\frac{3}{l} \right) + \overline{EAD}_1(0) + \Delta \overline{EAD}_1 \left( -\frac{4}{l^2} \right)$$

$$K_{3 \times 12} = K_{12 \times 3} = -\overline{EA\eta_a} \left( -\frac{4}{l^2} \right) - \Delta \overline{EA\eta_a} \left( -\frac{1}{l} \right)$$

$$K_{3 \times 13} = K_{13 \times 3} = -\overline{EA\eta_a} \left( \frac{8}{l^2} \right) - \Delta \overline{EA\eta_a} \left( \frac{4}{l} \right)$$

$$K_{3 \times 14} = K_{14 \times 3} = -\overline{EA\eta_a} \left( -\frac{4}{l^2} \right) - \Delta \overline{EA\eta_a} \left( -\frac{3}{l} \right)$$

$$K_{4\times 4} = \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\eta\zeta}} \sin \theta_G \right) (3)$$

$$K_{4\times 5} = K_{5\times 4} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right)$$

$$K_{4\times 6} = K_{6\times 4} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) (1)$$

$$K_{4\times 7} = K_{7\times 4} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( -\frac{4}{l} \right)$$

$$K_{4\times 8} = K_{8\times 4} = \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right) (3)$$

$$K_{4\times 9} = K_{9\times 4} = \tau_0 \overline{EAD'_1} \left( \frac{1}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{1}{3} \right) + \overline{EAD'_1} \left( \frac{4}{l^2} \right) + \Delta \overline{EAD'_1} \left( \frac{4}{l} \right)$$

$$K_{4\times 10} = K_{10\times 4} = \tau_0 \overline{EAD'_1} \left( -\frac{4}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( -\frac{8}{3} \right) + \overline{EAD'_1} \left( -\frac{8}{l^2} \right) + \Delta \overline{EAD'_1} \left( -\frac{8}{l} \right)$$

$$K_{4\times 11} = K_{11\times 4} = \tau_0 \overline{EAD'_1} \left( \frac{3}{l} \right) + \Delta \tau_0 \overline{EAD'_1} \left( \frac{7}{3} \right) + \overline{EAD'_1} \left( \frac{4}{l^2} \right) + \Delta \overline{EAD'_1} \left( \frac{4}{l} \right)$$

$$K_{4\times 12} = K_{12\times 4} = -\overline{EA\eta_a} \left( \frac{1}{l} \right) - \Delta \overline{EA\eta_a} \left( \frac{1}{3} \right)$$

$$K_{4\times 13} = K_{13\times 4} = -\overline{EA\eta_a} \left( -\frac{4}{l} \right) - \Delta \overline{EA\eta_a} \left( -\frac{8}{3} \right)$$

$$K_{4\times 14} = K_{14\times 4} = -\overline{EA\eta_a} \left( \frac{3}{l} \right) - \Delta \overline{EA\eta_a} \left( \frac{7}{3} \right)$$

$$K_{5\times 5} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{5\times 6} = K_{6\times 5} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right)$$

$$K_{5\times 7} = K_{7\times 5} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right)$$

$$K_{5\times 8} = K_{8\times 5} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right)$$

$$K_{5\times 9} = K_{9\times 5} = \tau_0 \overline{EAD'_2} \left( \frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( \frac{1}{l} \right) + \overline{EAD_2}(0) + \Delta \overline{EAD_2} \left( \frac{4}{l^2} \right)$$

$$K_{5\times 10} = K_{10\times 5} = \tau_0 \overline{EAD'_2} \left( -\frac{8}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( -\frac{4}{l} \right) + \overline{EAD_2}(0) + \Delta \overline{EAD_2} \left( -\frac{8}{l^2} \right)$$

$$K_{5\times 11} = K_{11\times 5} = \tau_0 \overline{EAD'_2} \left( \frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( \frac{3}{l} \right) + \overline{EAD_2}(0) + \Delta \overline{EAD_2} \left( \frac{4}{l^2} \right)$$

$$K_{5\times 12} = K_{12\times 5} = -\overline{EA\zeta_a} \left( \frac{4}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( \frac{1}{l} \right)$$

$$K_{5\times 13} = K_{13\times 5} = -\overline{EA\zeta_a} \left( -\frac{8}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( -\frac{4}{l} \right)$$

$$K_{5\times 14} = K_{14\times 5} = -\overline{EA\zeta_a} \left( \frac{4}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( \frac{3}{l} \right)$$

$$K_{6 \times 6} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) (1)$$

$$K_{6 \times 7} = K_{7 \times 6} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{2}{l} \right)$$

$$K_{6 \times 8} = K_{8 \times 6} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{2}{l} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) (1)$$

$$K_{6 \times 9} = K_{9 \times 6} = \tau_0 \overline{EAD'_2} \left( \frac{3}{l} \right) + \Delta \tau_0 \overline{EAD'_2} \left( \frac{2}{3} \right) + \overline{EAD_2} \left( -\frac{4}{l^2} \right) + \Delta \overline{EAD_2} (0)$$

$$K_{6 \times 10} = K_{10 \times 6} = \tau_0 \overline{EAD'_2} \left( -\frac{4}{l} \right) + \Delta \tau_0 \overline{EAD'_2} \left( -\frac{4}{3} \right) + \overline{EAD_2} \left( \frac{8}{l^2} \right) + \Delta \overline{EAD_2} (0)$$

$$K_{6 \times 11} = K_{11 \times 6} = \tau_0 \overline{EAD'_2} \left( \frac{1}{l} \right) + \Delta \tau_0 \overline{EAD'_2} \left( \frac{2}{3} \right) + \overline{EAD_2} \left( -\frac{4}{l^2} \right) + \Delta \overline{EAD_2} (0)$$

$$K_{6 \times 12} = K_{12 \times 6} = -\overline{EA\zeta_a} \left( \frac{3}{l} \right) - \Delta \overline{EA\zeta_a} \left( \frac{2}{3} \right)$$

$$K_{6 \times 13} = K_{13 \times 6} = -\overline{EA\zeta_a} \left( -\frac{4}{l} \right) - \Delta \overline{EA\zeta_a} \left( -\frac{4}{3} \right)$$

$$K_{6 \times 14} = K_{14 \times 6} = -\overline{EA\zeta_a} \left( \frac{1}{l} \right) - \Delta \overline{EA\zeta_a} \left( \frac{2}{3} \right)$$

$$K_{7 \times 7} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{12}{l^3} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{6}{l^2} \right)$$

$$K_{7 \times 8} = K_{8 \times 7} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{6}{l^2} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( -\frac{4}{l} \right)$$

$$K_{7 \times 9} = K_{9 \times 7} = \tau_0 \overline{EAD'_2} \left( -\frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( -\frac{1}{l} \right) + \overline{EAD_2} (0) + \Delta \overline{EAD_2} \left( -\frac{4}{l^2} \right)$$

$$K_{7 \times 10} = K_{10 \times 7} = \tau_0 \overline{EAD'_2} \left( \frac{8}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( \frac{4}{l} \right) + \overline{EAD_2} (0) + \Delta \overline{EAD_2} \left( \frac{8}{l^2} \right)$$

$$K_{7 \times 11} = K_{11 \times 7} = \tau_0 \overline{EAD'_2} \left( -\frac{4}{l^2} \right) + \Delta \tau_0 \overline{EAD'_2} \left( -\frac{3}{l} \right) + \overline{EAD_2} (0) + \Delta \overline{EAD_2} \left( -\frac{4}{l^2} \right)$$

$$K_{7 \times 12} = K_{12 \times 7} = -\overline{EA\zeta_a} \left( -\frac{4}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( -\frac{1}{l} \right)$$

$$K_{7 \times 13} = K_{13 \times 7} = -\overline{EA\zeta_a} \left( -\frac{8}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( \frac{4}{l} \right)$$

$$K_{7 \times 14} = K_{14 \times 7} = -\overline{EA\zeta_a} \left( -\frac{4}{l^2} \right) - \Delta \overline{EA\zeta_a} \left( -\frac{3}{l} \right)$$

$$K_{8 \times 8} = \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) \left( \frac{4}{l} \right) + \Delta \left( \overline{EI_{\eta\eta}} \cos \theta_G + \overline{EI_{\zeta\eta}} \sin \theta_G \right) (3)$$

$$K_{8 \times 9} = K_{9 \times 8} = \tau_0 \overline{EAD}'_2 \left( \frac{1}{l} \right) + \Delta \tau_0 \overline{EAD}'_2 \left( \frac{1}{3} \right) + \overline{EAD}_2 \left( \frac{4}{l^2} \right) + \Delta \overline{EAD}_2 \left( \frac{4}{l} \right)$$

$$K_{8 \times 10} = K_{10 \times 8} = \tau_0 \overline{EAD}'_2 \left( -\frac{4}{l} \right) + \Delta \tau_0 \overline{EAD}'_2 \left( -\frac{8}{3} \right) + \overline{EAD}_2 \left( -\frac{8}{l^2} \right) + \Delta \overline{EAD}_2 \left( -\frac{8}{l} \right)$$

$$K_{8 \times 11} = K_{11 \times 8} = \tau_0 \overline{EAD}'_2 \left( \frac{3}{l} \right) + \Delta \tau_0 \overline{EAD}'_2 \left( \frac{7}{3} \right) + \overline{EAD}_2 \left( \frac{4}{l^2} \right) + \Delta \overline{EAD}_2 \left( \frac{4}{l} \right)$$

$$K_{8 \times 12} = K_{12 \times 8} = -\overline{EA\zeta}_a \left( \frac{1}{l} \right) - \Delta \overline{EA\zeta}_a \left( \frac{1}{3} \right)$$

$$K_{8 \times 13} = K_{13 \times 8} = -\overline{EA\zeta}_a \left( -\frac{4}{l} \right) - \Delta \overline{EA\zeta}_a \left( -\frac{8}{3} \right)$$

$$K_{8 \times 14} = K_{14 \times 8} = -\overline{EA\zeta}_a \left( \frac{3}{l} \right) - \Delta \overline{EA\zeta}_a \left( \frac{7}{3} \right)$$

$$K_{9 \times 9} = \left( GJ + \tau_0^2 EAD'_3 \right) \left( \frac{7}{3l} \right) + \Delta \left( GJ + \tau_0^2 EAD'_3 \right) \left( \frac{1}{2} \right) + \tau_0 EAD_5 \left( -\frac{4}{l^2} - \frac{4}{l^2} \right) +$$

$$\Delta \tau_0 EAD_5 \left( -\frac{2}{3l} - \frac{2}{3l} \right) + EAD_3 \left( \frac{16}{l^3} \right) + \Delta EAD_3 \left( \frac{8}{l^2} \right)$$

$$K_{9 \times 10} = K_{10 \times 9} = \left( GJ + \tau_0^2 EAD'_3 \right) \left( -\frac{8}{3l} \right) + \Delta \left( GJ + \tau_0^2 EAD'_3 \right) \left( -\frac{2}{3} \right) + \tau_0 EAD_5 \left( \frac{8}{l^2} + 0 \right) +$$

$$\Delta \tau_0 EAD_5 \left( \frac{4}{3l} - \frac{8}{3l} \right) + EAD_3 \left( -\frac{32}{l^3} \right) + \Delta EAD_3 \left( -\frac{16}{l^2} \right)$$

$$K_{9 \times 11} = K_{11 \times 9} = \left( GJ + \tau_0^2 EAD'_3 \right) \left( \frac{1}{3l} \right) + \Delta \left( GJ + \tau_0^2 EAD'_3 \right) \left( \frac{1}{6} \right) + \tau_0 EAD_5 \left( -\frac{4}{l^2} + \frac{4}{l^2} \right) +$$

$$\Delta \tau_0 EAD_5 \left( -\frac{2}{3l} + \frac{10}{3l} \right) + EAD_3 \left( \frac{16}{l^3} \right) + \Delta EAD_3 \left( \frac{8}{l^2} \right)$$

$$K_{9 \times 12} = K_{12 \times 9} = -\tau_0 EAD'_0 \left( \frac{7}{3l} \right) - \Delta \tau_0 EAD'_0 \left( \frac{1}{2} \right) - EAD_0 \left( -\frac{4}{l^2} \right) - \Delta EAD_0 \left( -\frac{2}{3l} \right)$$

$$K_{9 \times 13} = K_{13 \times 9} = -\tau_0 EAD'_0 \left( -\frac{8}{3l} \right) - \Delta \tau_0 EAD'_0 \left( -\frac{2}{3} \right) - EAD_0 \left( 0 \right) - \Delta EAD_0 \left( -\frac{8}{3l} \right)$$

$$K_{9 \times 14} = K_{14 \times 9} = -\tau_0 EAD'_0 \left( \frac{1}{3l} \right) - \Delta \tau_0 EAD'_0 \left( \frac{1}{6} \right) - EAD_0 \left( \frac{4}{l^2} \right) - \Delta EAD_0 \left( \frac{10}{3l} \right)$$

$$K_{10 \times 10} = K_{10 \times 10} = (GJ + \tau_0^2 EAD'_3) \left( \frac{16}{3l} \right) + \Delta (GJ + \tau_0^2 EAD'_3) \left( \frac{8}{3} \right) + \tau_0 EAD_s (0 + 0) \\ + \Delta \tau_0 EAD_s \left( \frac{16}{3l} + \frac{16}{3l} \right) + EAD_3 \left( \frac{64}{l^3} \right) + \Delta EAD_3 \left( \frac{32}{l^2} \right)$$

$$K_{10 \times 11} = K_{11 \times 10} = (GJ + \tau_0^2 EAD'_3) \left( -\frac{8}{3l} \right) + \Delta (GJ + \tau_0^2 EAD'_3) (-2) + \tau_0 EAD_s \left( 0 - \frac{8}{l^2} \right) \\ + \Delta \tau_0 EAD_s \left( -\frac{8}{3l} - \frac{20}{3l} \right) + EAD_3 \left( -\frac{32}{l^3} \right) + \Delta EAD_3 \left( -\frac{16}{l^2} \right)$$

$$K_{10 \times 12} = K_{12 \times 10} = -\tau_0 EAD'_0 \left( -\frac{8}{3l} \right) - \Delta \tau_0 EAD'_0 \left( -\frac{2}{3} \right) - EAD_0 \left( \frac{8}{l^2} \right) - \Delta EAD_0 \left( \frac{4}{3l} \right)$$

$$K_{10 \times 13} = K_{13 \times 10} = -\tau_0 EAD'_0 \left( \frac{16}{3l} \right) - \Delta \tau_0 EAD'_0 \left( \frac{8}{3} \right) - EAD_0 (0) - \Delta EAD_0 \left( \frac{16}{3l} \right)$$

$$K_{10 \times 14} = K_{14 \times 10} = -\tau_0 EAD'_0 \left( -\frac{8}{3l} \right) - \Delta \tau_0 EAD'_0 (-2) - EAD_0 \left( -\frac{8}{l^2} \right) - \Delta EAD_0 \left( -\frac{20}{3l} \right)$$

$$K_{11 \times 11} = K_{11 \times 11} = (GJ + \tau_0^2 EAD'_3) \left( \frac{7}{3l} \right) + \Delta (GJ + \tau_0^2 EAD'_3) \left( \frac{11}{6} \right) + \tau_0 EAD_s \left( \frac{4}{l^2} + \frac{4}{l^2} \right) \\ + \Delta \tau_0 EAD_s \left( \frac{10}{3l} + \frac{10}{3l} \right) + EAD_3 \left( \frac{16}{l^3} \right) + \Delta EAD_3 \left( \frac{8}{l^2} \right)$$

$$K_{11 \times 12} = K_{12 \times 11} = -\tau_0 EAD'_0 \left( \frac{1}{3l} \right) - \Delta \tau_0 EAD'_0 \left( \frac{1}{6} \right) - EAD_0 \left( -\frac{4}{l^2} \right) - \Delta EAD_0 \left( -\frac{2}{3l} \right)$$

$$K_{11 \times 13} = K_{13 \times 11} = -\tau_0 EAD'_0 \left( -\frac{8}{3l} \right) - \Delta \tau_0 EAD'_0 (-2) - EAD_0 (0) - \Delta EAD_0 \left( -\frac{8}{3l} \right)$$

$$K_{11 \times 14} = K_{14 \times 11} = -\tau_0 EAD'_0 \left( \frac{7}{3l} \right) - \Delta \tau_0 EAD'_0 \left( \frac{11}{6} \right) - EAD_0 \left( \frac{4}{l^2} \right) - \Delta EAD_0 \left( \frac{10}{3l} \right)$$

$$K_{12 \times 12} = EA \left( \frac{7}{3l} \right) + \Delta EA \left( \frac{1}{2} \right)$$

$$K_{12 \times 13} = K_{13 \times 12} = EA \left( -\frac{8}{3l} \right) + \Delta EA \left( -\frac{2}{3} \right)$$

$$\begin{aligned}
K_{12 \times 14} &= K_{14 \times 12} = EA \left( \frac{1}{3l} \right) + \Delta EA \left( \frac{1}{6} \right) \\
K_{13 \times 13} &= EA \left( \frac{16}{3l} \right) + \Delta EA \left( \frac{8}{3} \right) \\
K_{13 \times 14} &= K_{14 \times 13} = EA \left( -\frac{8}{3l} \right) + \Delta EA (-2) \\
K_{14 \times 14} &= EA \left( \frac{7}{3l} \right) + \Delta EA \left( \frac{11}{6} \right)
\end{aligned} \tag{7.5}$$

## 7.2 Nonlinear Stiffness Vector

The nonlinear stiffness vector  $\{F^E\}$  is given by the following sub-vectors:

$$\{F^E\} = \begin{Bmatrix} \{F_1^E\} \\ \{F_2^E\} \\ \{F_3^E\} \\ \{F_4^E\} \end{Bmatrix} \tag{7.6}$$

$$\begin{aligned}
\{F_1^E\} &= \int_0^l \left\{ \left\{ v_x \left[ EA \left( u_x + 0.5(v_x^2 + w_x^2) \right) - \phi_{xx} EAD_0 + 0.5\phi_x^2 EAC_0 - \phi_x \tau_0 EAD'_0 \right. \right. \right. \\
&\quad \left. \left. \left. - w_{xx} \left( \overline{EA\zeta_a} + \phi \overline{EA\eta_a} \right) - v_{xx} \left( \overline{EA\eta_a} - \phi \overline{EA\zeta_a} \right) \right] \right. \right. \\
&\quad \left. \left. + \sin \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \left[ \phi_x (-GJ_0 - GJ_1) - \phi_0 GJ_0 \right] \right\} \{ \phi'_c \} \right. \\
&\quad \left. + \left\{ \cos \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) \left[ \phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0 \right] \right. \right. \\
&\quad \left. \left. - 0.5(v_x^2 + w_x^2) \overline{EA\eta_a} - 0.5\phi_x^2 \overline{EAC_1} + \phi \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\zeta_a} \right. \right. \right. \\
&\quad \left. \left. \left. - \phi_x \tau_0 \overline{EAD'_2} + 0.5\phi_x^2 \overline{EAC_2} - w_{xx} \left[ \left( \overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}} \right) \sin \theta_G + \left( \overline{EI_{\eta\eta}} - \overline{EI_{\zeta\zeta}} \right) \cos \theta_G \right] \right. \right. \\
&\quad \left. \left. \left. - v_{xx} \left[ \left( \overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}} \right) \sin \theta_G + \left( \overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}} \right) \cos \theta_G \right] \right\} \{ \phi''_c \} \right\} dx
\end{aligned}$$

$$\begin{aligned}
\{F_2^E\} = & \int_0^{l_x} \left\{ \left\{ w_x \left[ EA \left( u_x + 0.5(v_x^2 + w_x^2) \right) - \phi_{xx} EAD_0 + 0.5\phi_x^2 EAC_0 - \phi_x \tau_0 EAD'_0 \right. \right. \right. \\
& - w_{xx} \left( \overline{EA\zeta_a} + \phi \overline{EA\eta_a} \right) - v_{xx} \left( \overline{EA\eta_a} - \phi \overline{EA\zeta_a} \right) \\
& \left. \left. \left. + \cos \theta_G (v_{xx} \cos \theta_G + w_{xx} \sin \theta_G) \left[ \phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0 \right] \right\} \{ \phi'_c \} \right. \right. \\
& + \left\{ \sin \theta_G (w_x \cos \theta_G - v_x \sin \theta_G) \left[ \phi_x (GJ_0 + GJ_1) + \phi_0 GJ_0 \right] \right. \\
& - 0.5(v_x^2 + w_x^2) \overline{EA\zeta_a} - 0.5\phi_x^2 \overline{EAC_2} + \phi \left[ -\phi_{xx} \overline{EAD_1} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\eta_a} \right. \\
& - \phi_x \tau_0 \overline{EAD'_1} + 0.5\phi_x^2 \overline{EAC_1} - w_{xx} \left[ (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \cos \theta_G + (-\overline{EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G \right] \\
& \left. \left. \left. - v_{xx} \left[ (\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G \right] \right\} \{ \phi''_c \} \right\} dx
\end{aligned}$$

$$\begin{aligned}
\{F_3^E\} = & \int_0^{l_x} \left\{ v_{xx} \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\zeta_a} - \phi_x \tau_0 \overline{EAD'_2} + 0.5\phi_x^2 \overline{EAC_2} \right] \right. \\
& - w_{xx} \left[ \overline{EI_{\zeta\eta}} \sin \theta_G + \overline{EI_{\eta\eta}} \cos \theta_G \right] - v_{xx} \left[ \overline{EI_{\zeta\zeta}} \sin \theta_G + \overline{EI_{\eta\zeta}} \cos \theta_G \right] \\
& + \phi \left[ -\phi_{xx} \overline{EAD_1} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\eta_a} - \phi_x \tau_0 \overline{EAD'_1} + 0.5\phi_x^2 \overline{EAC_1} \right] \\
& - w_{xx} [(\overline{EI_{\zeta\eta}} + \overline{EI_{\zeta\eta}}) \cos \theta_G + (-\overline{EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G] \\
& - v_{xx} [(\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G] \\
& + w_{xx} [-\phi_{xx} \overline{EAD_1} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\eta_a} - \phi_x \tau_0 \overline{EAD'_1} + 0.5\phi_x^2 \overline{EAC_1}] \\
& + w_{xx} \left[ \overline{EI_{\zeta\eta}} \cos \theta_G - \overline{EI_{\eta\eta}} \sin \theta_G \right] + v_{xx} \left[ \overline{EI_{\zeta\zeta}} \cos \theta_G - \overline{EI_{\zeta\eta}} \sin \theta_G \right] \\
& + \phi \left[ -\phi_{xx} \overline{EAD_2} + (u_x + 0.5(v_x^2 + w_x^2)) \overline{EA\zeta_a} - \phi_x \tau_0 \overline{EAD'_2} + 0.5\phi_x^2 \overline{EAC_2} \right] \\
& - w_{xx} \left[ (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \cos \theta_G + (-\overline{EI_{\eta\eta}} + \overline{EI_{\zeta\zeta}}) \sin \theta_G \right] \\
& - v_{xx} \left[ (\overline{EI_{\zeta\zeta}} - \overline{EI_{\eta\eta}}) \cos \theta_G - (\overline{EI_{\zeta\eta}} + \overline{EI_{\eta\zeta}}) \sin \theta_G \right] \} \{ \phi_q \} \\
& + \left\{ -\tau_o \left[ 0.5EAD'_0(v_x^2 + w_x^2) + 0.5EAD'_4\phi_x^2 - \phi (w_{xx} \overline{EAD'_1} - v_{xx} \overline{EAD'_2}) \right] \right. \\
& + \phi_x \left[ -\phi_{xx} EAD_4 + (u_x + 0.5(v_x^2 + w_x^2)) EAC_0 \right. \\
& \left. \left. - \phi_x \tau_o EAD'_4 + 0.5\phi_x^2 EAC_3 - w_{xx} (\overline{EAC_2} + \phi \overline{EAC_1}) - v_{xx} (\overline{EAC_1} - \phi \overline{EAC_2}) \right] \right\} \{ \phi'_q \}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -0.5EAD_0(v_x^2 + w_x^2) - 0.5EAD_4\phi_x^2 + \phi(w_{xx}\overline{EAD}_1 - v_{xx}\overline{EAD}_2) \right\} \{\phi''_q\} dx \\
\{F_4^E\} & = \int_0^1 \left\{ 0.5EA(v_x^2 + w_x^2) + 0.5\phi_x^2EAC_0 - \phi(w_{xx}\overline{EA\eta_a} + v_{xx}\overline{EA\zeta_a}) \right\} \{\phi'_q\} dx
\end{aligned} \quad (7.7)$$

The underlined terms in the above sub-vectors are those, which are associated with the axial strain  $(u_x + 0.5(v_x^2 + w_x^2))$  at the elastic axis of the blade. These nonlinear terms are modified to a set of linear terms by substituting the axial stress by the axial inertial force. The description of the substitution approach is given in the next section.

### 7.3 Linearization of Nonlinear Terms Associated With The Axial Strain At The Elastic Axis

The equation of motion corresponding to the axial degree of freedom can be written symbolically as:

$$\frac{E_0}{m\Omega^2} \left[ -\overline{V}_x \right] - \overline{Z}_u = 0 \quad (7.8)$$

The axial stress resultant can be written as

$$\overline{V}_x = EA \left[ u_x + 0.5(v_x^2 + w_x^2) \right] + \overline{F} \quad (7.9)$$

Where  $\overline{F}$  contains all the additional higher order terms which can be neglected. The simplified expression for  $\overline{V}_x$  can be written as

$$\overline{V}_x \approx EA \left[ u_x + 0.5(v_x^2 + w_x^2) \right] \quad (7.10)$$

Neglecting all the higher order terms and also the time derivative terms, the distributed inertia force  $\overline{Z}_u$  can be written as:

$$\bar{Z}_u = C_1 \left[ \sum_{i=1}^{n-1} (l_e)_i + x_k \right] + C_2 \quad (7.11)$$

where

$$C_1 = \left\{ 1 - (\beta_s \cos \theta_l + \beta_d \sin \theta_l) \right\} \cos \theta_l^2 \cos \Lambda_s \cos \Lambda_a$$

$$C_2 = \left\{ (e_1 + e_2) - a (\beta_s \cos \theta_l + \beta_d \sin \theta_l) \right\} \cos \theta_l^2 \cos \Lambda_s \cos \Lambda_a$$

Where the symbols have the usual meanings.(are given in chapter 5)

Integrating eq.7.8.

$$\bar{V}_x = \int_{nl_e - x_k}^l \bar{Z}_u dx \quad (7.12)$$

From the above we see that  $\bar{V}_x$  can be written as:

$$\bar{V}_x = f(\xi) = a_1 \xi^2 + a_2 \xi + a_3 \quad (7.13)$$

where,

$$\xi = \frac{x_k}{l}$$

$a_1, a_2, a_3$  are given as:

$$a_1 = \frac{m\Omega^2}{E_0} \left[ -\frac{1}{2} C_1 l_e^2 \right]$$

$$a_2 = \frac{m\Omega^2}{E_0} \left[ -C_1 \sum_{i=1}^{n-1} (l_e)_i - C_2 \right] l_e \quad (7.14)$$

$$a_3 = \frac{m\Omega^2}{E_0} \left[ \frac{1}{2} C_1 \left\{ \sum_{i=1}^{n-1} (l_e)_i + \sum_{i=1}^{n-1} (l_e)_i + c_2 \right\} \right] \left[ \sum_{i=1}^N (l_e)_i - \left[ \sum_{i=1}^{n-1} (l_e)_i \right] \right]$$

so finally we get the axial strain terms as a quadratic polynomial:

$$[u_x + 0.5(v_x^2 + w_x^2)] = \frac{f(\xi)}{EA} \quad (7.15)$$

Now linearize the underlined terms using the above relation to substitute the axial strain term. The following additional integrals apart from those already listed earlier(6.8)

$$\int_0^l \xi^2 \{\phi'_c\} \{\phi'_c\}^T dx = \begin{bmatrix} \frac{12}{35l} & \frac{1}{14} & \frac{-12}{35l} & \frac{-1}{35} \\ \frac{1}{14} & \frac{2l}{105} & \frac{-1}{14} & \frac{-l}{70} \\ \frac{-12}{35l} & \frac{-1}{14} & \frac{12}{35l} & \frac{1}{35} \\ \frac{-1}{35} & \frac{-l}{70} & \frac{1}{35} & \frac{3l}{35} \end{bmatrix}$$

$$\int_0^l \xi (\phi'_c) (\phi'_c)^T dx = \begin{bmatrix} \frac{3}{5l} & \frac{1}{10} & \frac{-3}{5l} & 0 \\ \frac{1}{10} & \frac{l}{30} & \frac{-1}{10} & \frac{-l}{60} \\ \frac{-3}{5l} & \frac{-1}{10} & \frac{3}{5l} & 0 \\ 0 & \frac{-l}{60} & 0 & \frac{l}{10} \end{bmatrix}$$

$$\int_0^l (\phi_c') (\phi_c')^T dx = \begin{bmatrix} \frac{6}{5l} & \frac{1}{10} & \frac{-6}{5l} & \frac{1}{10} \\ \frac{1}{10} & \frac{2l}{15} & \frac{-1}{10} & \frac{-l}{30} \\ \frac{-6}{5l} & \frac{-1}{10} & \frac{6}{5l} & \frac{-1}{10} \\ \frac{1}{10} & \frac{-l}{30} & \frac{-1}{10} & \frac{2l}{15} \end{bmatrix}$$

$$\int_0^l \xi^2 \{\phi_c'\} \{\phi_c'\}^T dx = \begin{bmatrix} \frac{1}{15l} & \frac{-2}{5l} & \frac{1}{15l} \\ \frac{-2}{15l} & \frac{32}{15l} & \frac{-26}{15l} \\ \frac{1}{5l} & \frac{-26}{15l} & \frac{23}{15l} \end{bmatrix}$$

$$\int_0^l \xi (\phi_q') (\phi_q')^T dx = \begin{bmatrix} \frac{1}{2l} & \frac{-2}{3l} & \frac{1}{6l} \\ \frac{-2}{3l} & \frac{8}{3l} & \frac{-2}{l} \\ \frac{1}{6l} & \frac{-2}{l} & \frac{11}{6l} \end{bmatrix}$$

$$\int (\phi_q') (\phi_q')^T dx = \begin{bmatrix} \frac{7}{3l} & \frac{-8}{3l} & \frac{1}{3l} \\ \frac{-8}{3l} & \frac{16}{3l} & \frac{-8}{3l} \\ \frac{1}{3l} & \frac{-8}{3l} & \frac{7}{3l} \end{bmatrix} \quad (7.16)$$

From the underlined portion of the nonlinear stiffness vector, following was obtained:

$$\int_0^l \left( \delta v_x [a_1 \xi^2 + a_2 \xi + a_3] v_x + \delta w_x [a_1 \xi^2 + a_2 \xi + a_3] w_x + \delta \phi_x [a_1 \xi^2 + a_2 \xi + a_3] \phi_x \right) dx \quad (7.17)$$

From the above we get the following additional terms which to be added to the elemental stiffness matrix:

$$K_{1 \times 1} = \frac{12}{35l_e} a_1 + \frac{3}{5l_e} a_2 + \frac{6}{5l_e} a_3$$

$$K_{1 \times 2} = K_{2 \times 1} = \frac{1}{14} a_1 + \frac{1}{10} a_2 + \frac{1}{10} a_3$$

$$K_{1 \times 3} = K_{3 \times 1} = -\frac{12}{35l_e} a_1 - \frac{3}{5l_e} a_2 - \frac{6}{5l_e} a_3$$

$$K_{1 \times 4} = K_{4 \times 1} = -\frac{1}{35} a_1 + \frac{1}{10} a_3$$

$$K_{2 \times 2} = \frac{2l_e}{105l_e} a_1 + \frac{l_e}{30} a_2 + \frac{2l_e}{15} a_3$$

$$K_{2 \times 3} = K_{3 \times 2} = -\frac{1}{14} a_1 - \frac{1}{10} a_2 - \frac{1}{10} a_3$$

$$K_{2 \times 4} = K_{4 \times 2} = -\frac{l_e}{70} a_1 - \frac{l_e}{60} a_2 - \frac{l_e}{30} a_3$$

$$K_{3 \times 3} = \frac{12}{35l_e} a_1 + \frac{3}{5l_e} a_2 + \frac{6}{5l_e} a_3$$

$$K_{3 \times 4} = K_{4 \times 3} = \frac{1}{35} a_1 - \frac{1}{10} a_3$$

$$K_{4 \times 4} = \frac{3l_e}{35} a_1 + \frac{l_e}{10} a_2 + \frac{2l_e}{15} a_3$$

$$\begin{aligned}
K_{5 \times 5} &= K_{1 \times 1} \\
K_{5 \times 6} &= K_{6 \times 5} = K_{1 \times 2} \\
K_{5 \times 7} &= K_{7 \times 5} = K_{1 \times 3} \\
K_{5 \times 8} &= K_{8 \times 5} = K_{1 \times 4} \\
K_{6 \times 6} &= K_{2 \times 2} \\
K_{6 \times 7} &= K_{7 \times 6} = K_{2 \times 3} \\
K_{6 \times 8} &= K_{8 \times 6} = K_{2 \times 4} \\
K_{7 \times 7} &= K_{3 \times 3} \\
K_{7 \times 8} &= K_{8 \times 7} = K_{3 \times 4} \\
K_{8 \times 8} &= K_{4 \times 4}
\end{aligned}$$

$$\begin{aligned}
K_{9 \times 9} &= \frac{EAC_0}{EA} \left[ \frac{1}{5l_e} a_1 + \frac{1}{3l_e} a_2 + \frac{7}{3l_e} a_3 \right] \\
K_{9 \times 10} &= K_{10 \times 9} = \frac{EAC_0}{EA} \left[ -\frac{2}{5l_e} a_1 - \frac{2}{3l_e} a_2 - \frac{8}{3l_e} a_3 \right] \\
K_{9 \times 11} &= K_{11 \times 9} = \frac{EAC_0}{EA} \left[ \frac{1}{5l_e} a_1 + \frac{1}{6l_e} a_2 + \frac{1}{3l_e} a_3 \right] \\
K_{10 \times 10} &= \frac{EAC_0}{EA} \left[ \frac{32}{15l_e} a_1 + \frac{8}{3l_e} a_2 + \frac{16}{3l_e} a_3 \right] \\
K_{10 \times 11} &= K_{11 \times 10} = \frac{EAC_0}{EA} \left[ -\frac{26}{15l_e} a_1 - \frac{2}{l_e} a_2 - \frac{8}{l_e} a_3 \right] \\
K_{11 \times 11} &= \frac{EAC_0}{EA} \left[ \frac{23}{15l_e} a_1 + \frac{11}{6l_e} a_2 + \frac{7}{3l_e} a_3 \right]
\end{aligned}$$

(7.18)

# Chapter 8

## Results and Discussion

The first step in any aero-elastic response and stability analysis is the evaluation of natural frequencies of the rotor blade. Using the inertial and structural model developed in this study, a structural dynamic analysis was performed. It may be noted that the inertial and the structural operators given in Eqs.6.6 and 7.1 respectively, are nonlinear. Since the structural dynamic analysis requires only linear terms, all the nonlinear terms are neglected. The corresponding linear equation for one beam finite element can be written as

$$[M]_i \{\ddot{q}\}_i + [K]_i \{q\}_i = 0 \quad (8.1)$$

Where  $[M]_i$  represents the mass matrix of  $i^{th}$  element, (given in Eq.6.6 and in Sec.6.2.1). The stiffness matrix  $[K]_i$  consists of three components. They are  $[K^{cf}]$  (given in Sec.6.2.3),  $[K^E]$  (given in Eq. 7.5) and  $[K^{E'}]$  (given in Eq. 7.18). The element stiffness matrix is given as:

$$[K]_i = [K^{cf}] + [K^E] + [K^{E'}] \quad (8.2)$$

The element matrices are assembled to form the global finite element model for the rotor blade. For the tip element, the corresponding local to global transformation (Ref. 8) is performed to take in account the sweep and anhedral angles. The transformation matrix  $[\Lambda^c]$  is given in Appendix A. Imposing the root boundary conditions, the corresponding rows and columns from the global model are eliminated. The resulting matrix equation can be written as:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \quad (8.3)$$

Performing an eigen analysis, the natural frequencies of an un-damped rotating blade in vacuum can be evaluated

## 8.1 Validation

In order to validate the finite element blade model developed in this study, the results of the present analysis are compared (for certain specific cases) with those available in the literature. The data shown in Table 8.1 correspond to a uniform, untwisted straight blade for which uncoupled natural frequencies are available in the literature (Refs 1,3 and 6). The terminology soft-in-plate blade configuration indicates that the first non-dimensional rotating lag frequency is less than 1.

Using the data given Table 8.1, the uncoupled natural frequencies of the rotating blade are calculated. Natural frequencies are compared with those available in the literature (Refs 1,3, and 6) as shown in Table 8.2. The results of the present analysis are in excellent agreement with those available in literature. The natural frequencies that are given in the Table 8.2 is for N=20.

## 8.2 Practical data from the Industry

The code is designed to work with non-dimensional quantities. Hence, the practical data is made non-dimensional. For that, mass per unit length ( $m_0$ ) is taken as 8.45kg/m. Radius of the rotor blade ( $l$ ) is taken as 6.6 m. Speed of the rotor ( $\Omega_0$ ) is taken as 314 rpm.

The non-dimensionalization has been done in the following manner

$$\overline{EI} = \overline{GJ} = \frac{EI}{m_0 \Omega_0^2 l^4}$$

$$\overline{EA} = \frac{EA}{m_0 \Omega_0^2 l^2}$$

$$\overline{IM} = \frac{IM}{ml^2}$$

$$\overline{m} = \frac{m}{m_0}$$

The practical data obtained from the Industry contains non-uniform properties along the radial direction, and is given from 0 1136 (a non-dimensional value), as shown in Table 8.5. The variations in mass, mass moment of inertia and stiffness are shown in the Figs. 8.1-8.6. As a first attempt, two assumptions are made, i) the blade is fixed at 0.1136, and ii) no variation of properties occurs within the elements. The results are tabulated in Table 8.6. In the next attempt, a linear variation of properties within the elements is considered. The results for this case are tabulated in Table 8.6. The results and mode shapes hence generated are as shown in figs. 8.7-8.10.

The practical data also contains the root stiffness matrices from 0.0321 to 0 1136 (non-dimensional), for flap, lag and torsional modes, implying that the blade is fixed at 0.0321 (non-dimensional). The results hence simulated, are compared with the above-obtained results and are tabulated in Table 8.8. The effect of offset on frequencies is evident from the comparison. The mode shapes hence generated are as shown in figs. 8.11-8.14

Root stiffness matrix of the blade in lead-lag direction

$$\begin{bmatrix} 430.962 & -24.37 & -430.962 & -7.69 \\ -24.37 & 2.042 & 24.37 & -0.214 \\ -430.962 & 24.37 & 430.962 & 7.69 \\ -7.69 & -0.214 & 7.69 & 0.796 \end{bmatrix}$$

Root stiffness matrix of the blade in flap direction

$$\begin{bmatrix} 141.78 & 7.7102 & -141.78 & 3.8683 \\ 7.7102 & 0.466 & -7.7102 & 0.1629 \\ -141.78 & -7.7102 & 141.78 & -3.8683 \\ 3.8683 & 0.1629 & -3.8683 & 0.153 \end{bmatrix}$$

Root torsional stiffness matrix

$$(10^6) \begin{bmatrix} 6.951 & -8.056 & 1.105 \\ -8.056 & 9.756 & -1.698 \\ 1.105 & -1.698 & 0.595 \end{bmatrix}$$

### 8.3 Effect of root offset and Geometric pitch

The effect of root offset ( $e_1$ ) is analyzed for the uniform blade. The frequencies are proportionally increasing with the root offset. A convergence study is made by giving a high stiffness from 0 to 0.0321 (non-dimensional), to simulate a root offset. The results converged at high stiffness ( $\overline{EI} = 57680$ ) as shown in the Table 8.4.

The variation of geometric pitch is taken as  $12^\circ$  at 0.25 and  $4^\circ$  at the end tip (i.e., non-dimension length 1). The coupled lead-lag and flap modes are observed. The lead-lag and flap frequencies reduced due to the geometric pitch. The coupled modes are shown in the Figs 8.15 and 8.16.

Table 8.1 Input data for uniform soft-in-plane rotor blade

Uniform soft-in-plane rotor blade data (non dimensional properties)	
$Im_{\zeta\zeta}$	0.0004
$Im_{\eta\eta}$	0.000
$\theta_G$	0
$m$	1
$\beta_s$	0
$\beta_d$	0
$\beta_p$	0
$\theta_I$	0
GJ	0.001473
EA	20.0
$e_1$	0.0
$e_2$	0.0
a.	0.0
$C_0 = EAC_0 / EA$	0.00021036
$EI_{\zeta\zeta}$	0.0301
$EI_{\eta\eta}$	0.0106

Table 8.2 Natural frequencies of uniform soft-in-plane rotor blade

Mode	Uncoupled natural frequency				
	No. of elements (N = 20)	Ref. [2]	Ref.[1]	Ref.[3]	Ref.[6]
1 <sup>st</sup> Lag	0.7311	0.7311	0.7326	0.7317	0.732
2 <sup>nd</sup> Lag	4.4530	4.4531	4.4563	4.4825	-
3 <sup>rd</sup> Lag	11.2868	-	-	-	-
1 <sup>st</sup> Flap	1.1244	1.1251	1.1247	1.1245	1.125
2 <sup>nd</sup> Flap	3.4073	3.4266	3.4089	3.4073	-
3 <sup>rd</sup> Flap	7.6171	7.7154	7.6376	7.6218	-
1 <sup>st</sup> Torsion	3.2633	3.2633	3.2632	-	3.263
1 <sup>st</sup> Axial	6.9389	6.9389	6.9533	-	-

**Table 8.5** Industrial practical data of a helicopter blade

Radius	M	$IM_{\eta\eta}$	$IM_{\zeta\zeta}$	$EI_{\eta\eta}$	$EI_{\zeta\zeta}$	GJ
3.21E-02	2.46E+00	2.12E-05	6.82E-04	-	-	-
1.14E-01	1.44E+00	5.73E-06	6.41E-05	0.001990078	0.023189	0.002979
1.21E-01	1.35E+00	5.41E-06	5.32E-05	0.001863175	0.019093	0.002863
1.28E-01	1.27E+00	5.19E-06	4.40E-05	0.001892016	0.016094	0.002703
1.35E-01	1.21E+00	5.76E-06	3.91E-05	0.002220812	0.015805	0.00265
1.42E-01	1.14E+00	6.66E-06	5.53E-05	0.002549607	0.01194	0.002689
1.48E-01	1.07E+00	7.58E-06	3.12E-05	0.002884171	0.010095	0.002825
1.56E-01	1.02E+00	8.59E-06	2.70E-05	0.003189893	0.008306	0.002975
1.64E-01	9.93E-01	8.94E-06	2.50E-05	0.003143747	0.008018	0.003047
1.71E-01	9.61E-01	8.86E-06	2.44E-05	0.003091831	0.00773	0.003059
1.79E-01	9.28E-01	8.78E-06	2.38E-05	0.003034148	0.007441	0.00305
1.86E-01	8.97E-01	8.67E-06	2.31E-05	0.002976465	0.007153	0.003039
1.94E-01	8.66E-01	8.56E-06	2.27E-05	0.002901476	0.006864	0.003029
2.02E-01	8.85E-01	8.45E-06	2.23E-05	0.002838024	0.008249	0.003021
2.09E-01	9.05E-01	8.29E-06	5.49E-05	0.002878403	0.024227	0.003107
2.17E-01	9.24E-01	8.10E-06	1.10E-04	0.002918781	0.040263	0.00329
2.24E-01	9.43E-01	7.91E-06	1.65E-04	0.00295916	0.056241	0.003591
2.32E-01	9.80E-01	7.72E-06	2.20E-04	0.003016843	0.079603	0.003892
2.47E-01	1.03E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004479
2.65E-01	1.03E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
2.95E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
3.26E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
3.56E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
3.86E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
4.17E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
4.47E-01	1.02E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
4.77E-01	1.01E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
5.08E-01	1.01E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
5.38E-01	1.01E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
5.68E-01	1.01E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
5.98E-01	1.01E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
6.29E-01	1.00E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
6.59E-01	1.00E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
6.89E-01	1.00E+00	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
7.20E-01	9.99E-01	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
7.50E-01	9.96E-01	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
7.76E-01	9.95E-01	7.34E-06	3.23E-04	0.003057221	0.094601	0.004672
8.00E-01	9.94E-01	7.34E-06	3.23E-04	0.002861098	0.094601	0.004672
8.33E-01	9.64E-01	6.38E-06	3.18E-04	0.002388094	0.094024	0.004255
8.79E-01	9.27E-01	5.08E-06	3.12E-04	0.001897785	0.093447	0.003517
9.17E-01	8.95E-01	3.99E-06	3.10E-04	0.001626673	0.093447	0.002923
9.24E-01	8.54E-01	3.78E-06	3.07E-04	0.001511306	0.088832	0.002821
9.32E-01	8.14E-01	3.48E-06	2.72E-04	0.001222889	0.070374	0.002719
9.55E-01	5.94E-01	2.50E-06	1.68E-04	0.001176742	0.041994	0.001669
9.77E-01	4.18E-01	1.22E-06	7.72E-05	1.52E-03	0.012517	0.000886
1.00E+00	2.49E-01	2.72E-07	2.72E-06	1.52E-03	0.012517	0.002043

**Table 8.6** Comparison of natural frequencies for different assumptions in FE  
( $e_1 = 0.0321$ , No. of elements = 30)

Mode	Linear variation within the elements	No variation within the elements
1 <sup>st</sup> Lag	1.0341	1.0389
2 <sup>nd</sup> Lag	7.8315	7.8479
3 <sup>rd</sup> Lag	21.4294	21.461
1 <sup>st</sup> Flap	1.2296	1.2043
2 <sup>nd</sup> Flap	3.3896	3.3176
3 <sup>rd</sup> Flap	6.3276	6.2059
1 <sup>st</sup> Torsion	5.6264	5.5711
1 <sup>st</sup> Axial	8.3336	8.3967

**Table 8.7** Convergence of natural frequencies ( $e_1 = 0.0321$ )

Mode	No. of elements (N) = 20	No. of elements (N) = 30
1 <sup>st</sup> Lag	1.0343	1.0341
2 <sup>nd</sup> Lag	8.0399	7.8315
3 <sup>rd</sup> Lag	22.038	21.4294
1 <sup>st</sup> Flap	1.2289	1.2296
2 <sup>nd</sup> Flap	3.4134	3.3896
3 <sup>rd</sup> Flap	6.4059	6.3276
1 <sup>st</sup> Torsion	5.6269	5.6264
1 <sup>st</sup> Axial	8.3421	8.3336

**Table 8.8** Comparison of natural frequencies for different root offsets

Mode	Root offset $e_1 = 0.0321$	Root offset $e_1 = 0.1136$
1 <sup>st</sup> Lag	1.0343	1.269
2 <sup>nd</sup> Lag	8.0399	8.295
3 <sup>rd</sup> Lag	22.038	22.823
1 <sup>st</sup> Flap	1.2289	1.435
2 <sup>nd</sup> Flap	3.4134	4.092
3 <sup>rd</sup> Flap	6.4059	7.556
1 <sup>st</sup> Torsion	5.6269	9.1577
1 <sup>st</sup> Axial	8.3421	7.508

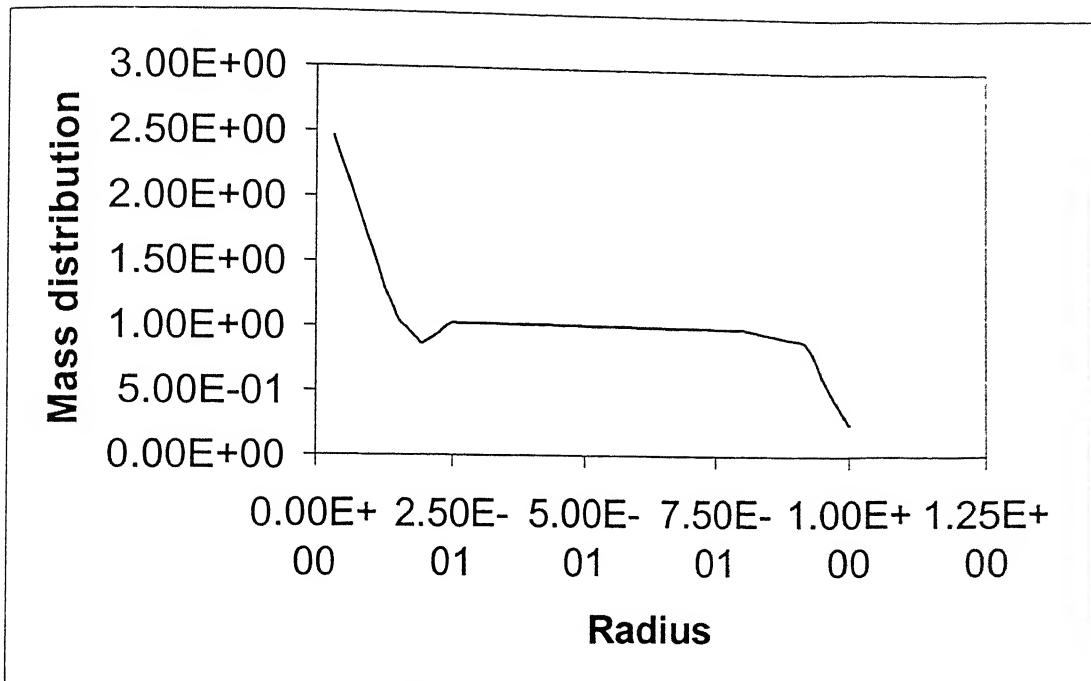


Figure 8.1 Distribution of mass along the radius

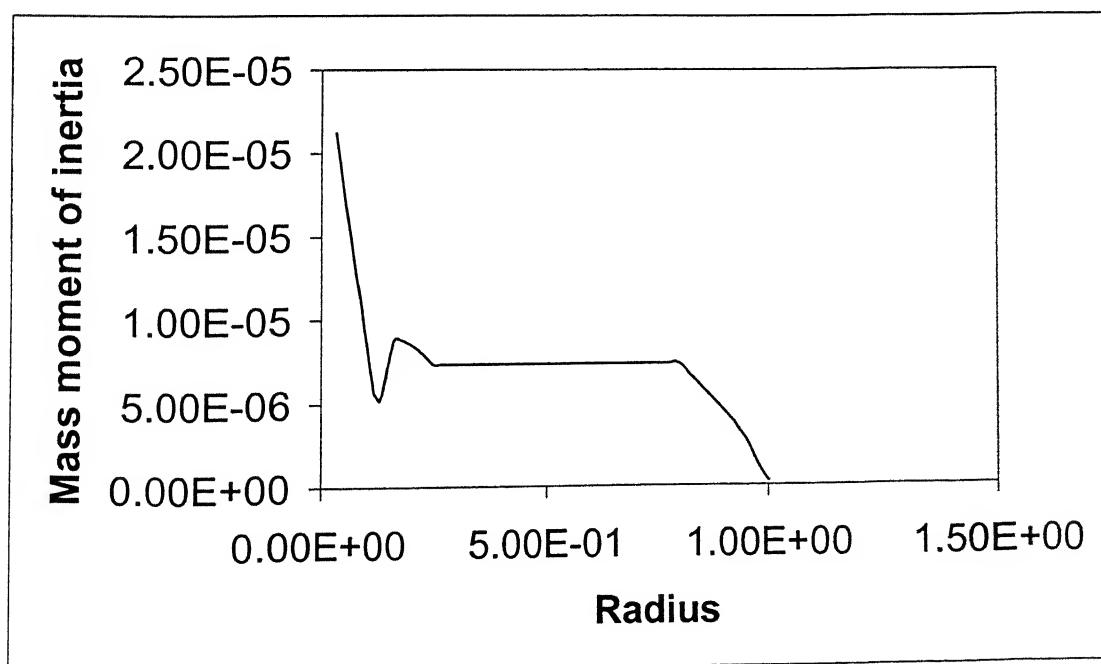


Figure 8.2 Distribution of mass moment of inertia in flap direction along the radius

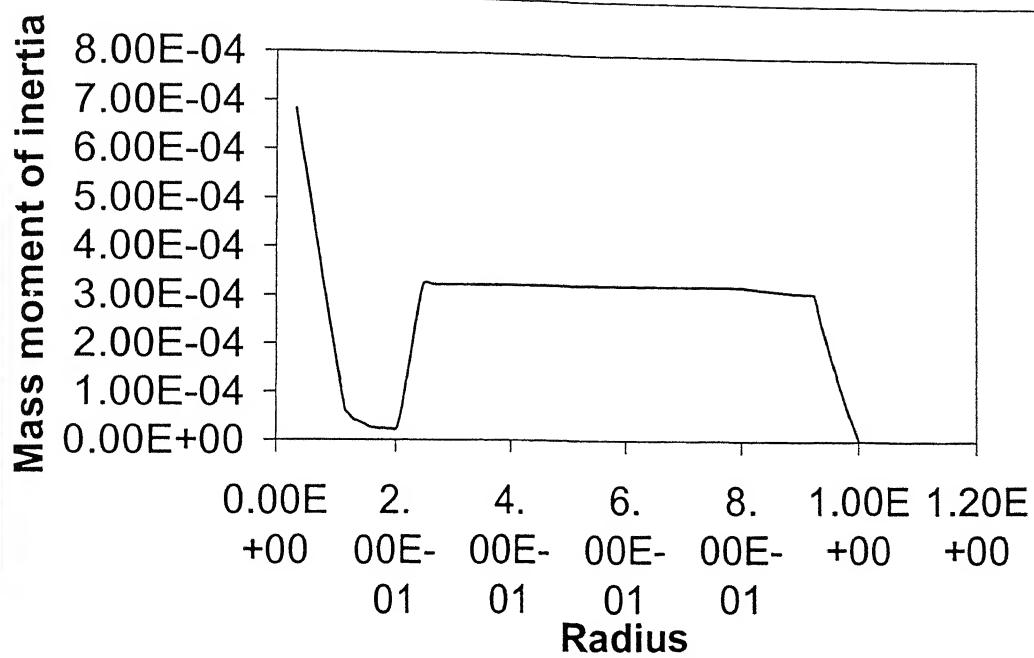


Figure 8.3 Distribution of mass moment of inertia in lead - lag direction along the radius

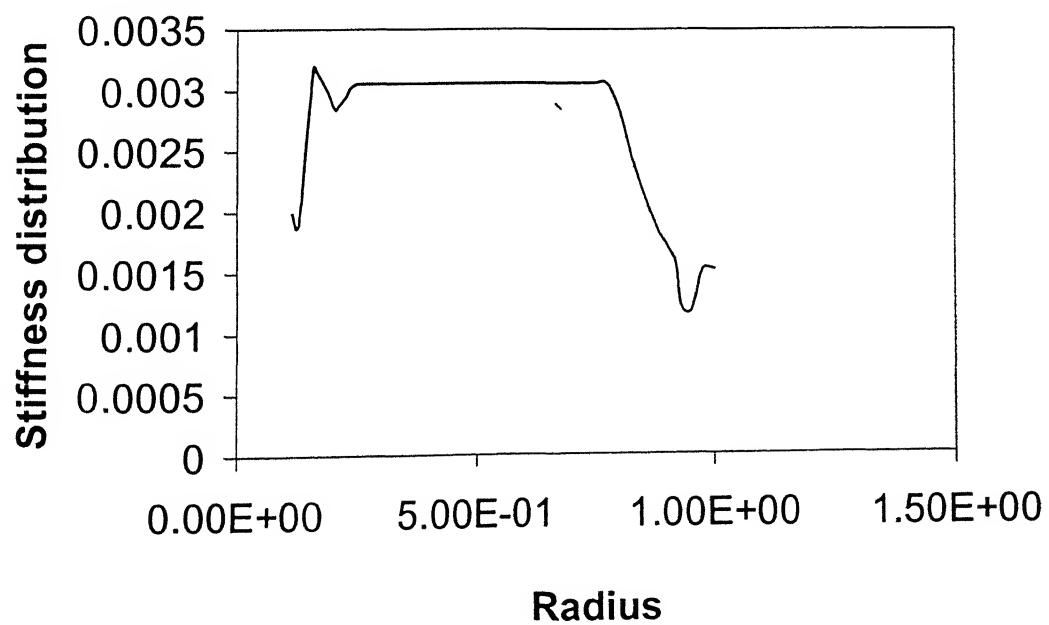


Figure 8.4 Distribution of flap wise bending stiffness along the radius

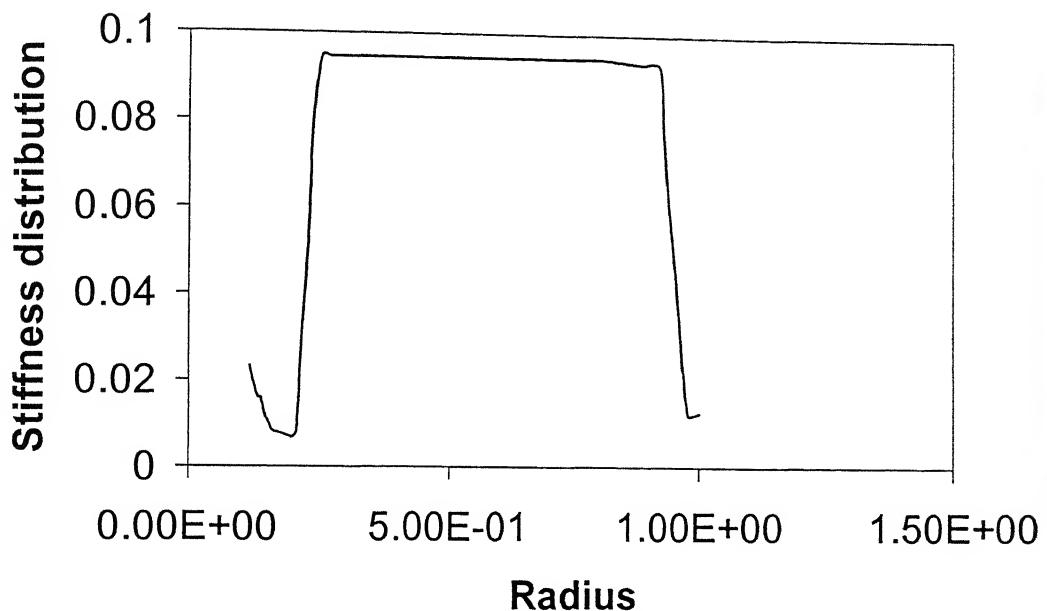


Figure 8.5 Distribution of lead - lag wise bending stiffness along the radius

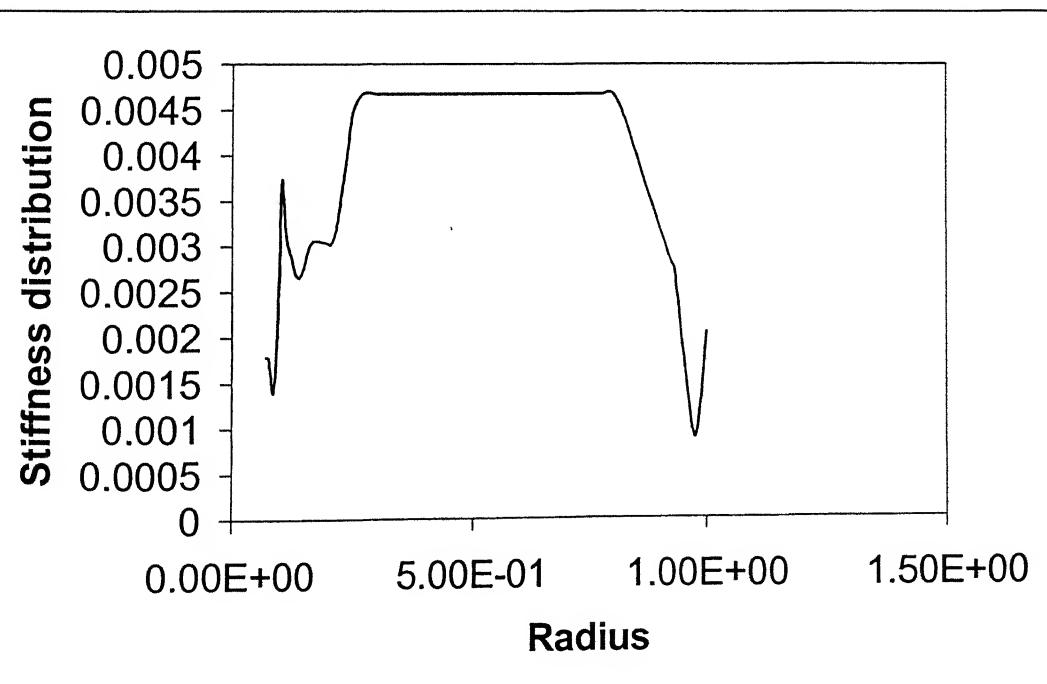


Figure 8.6 Torsional stiffness distribution along the radius

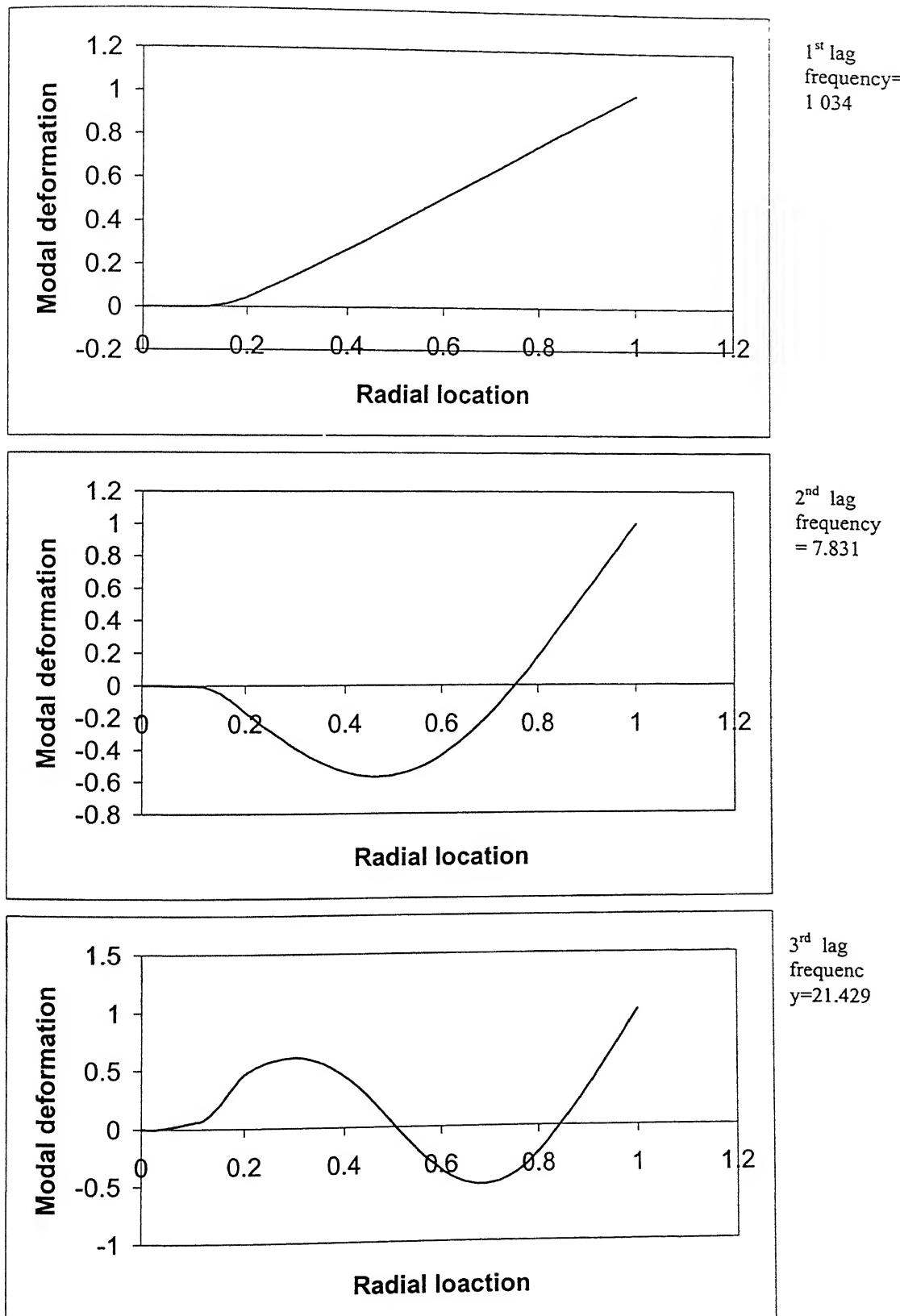


Figure 8.7 Mode shapes in lead-lag ( in plane bending ) mode

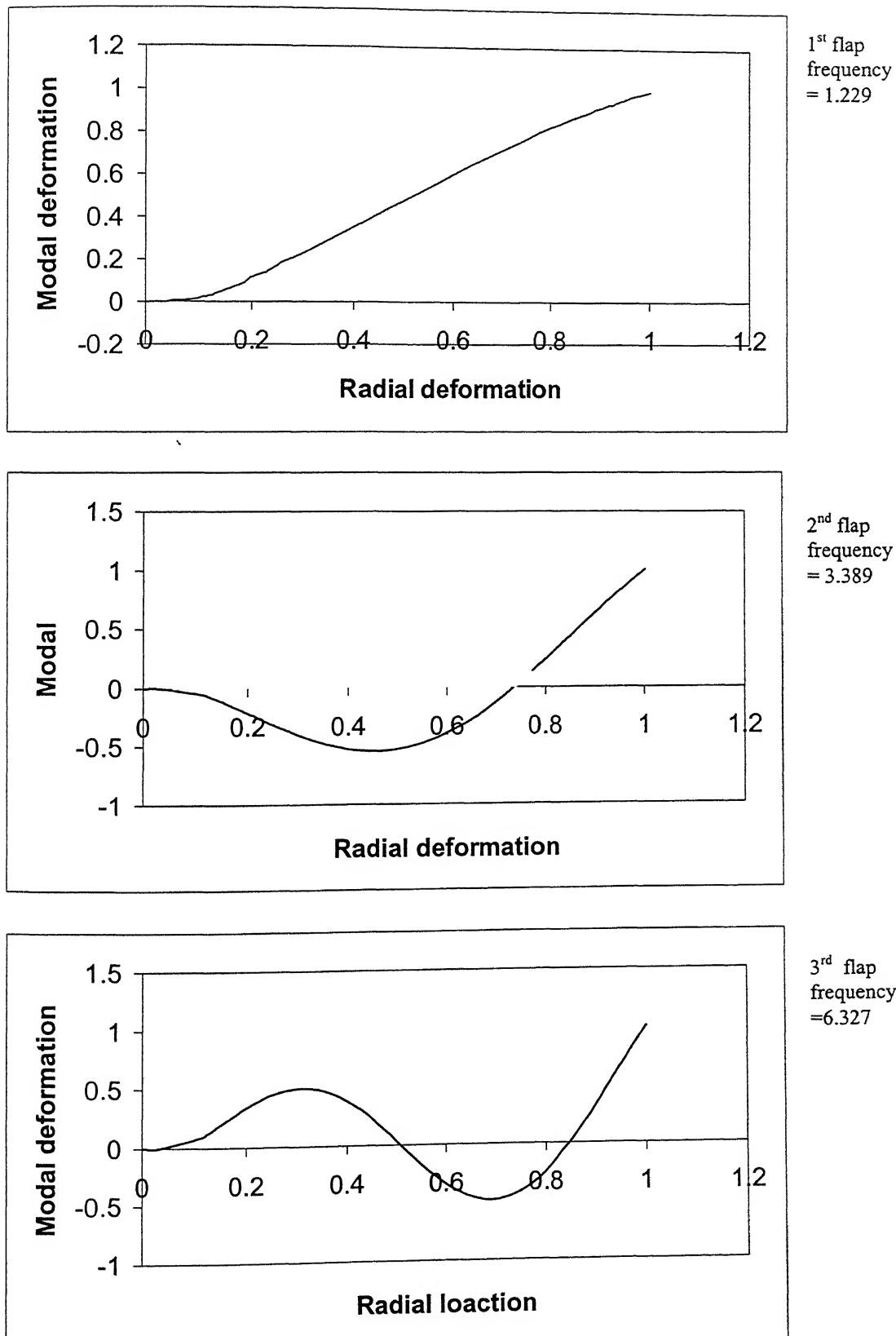
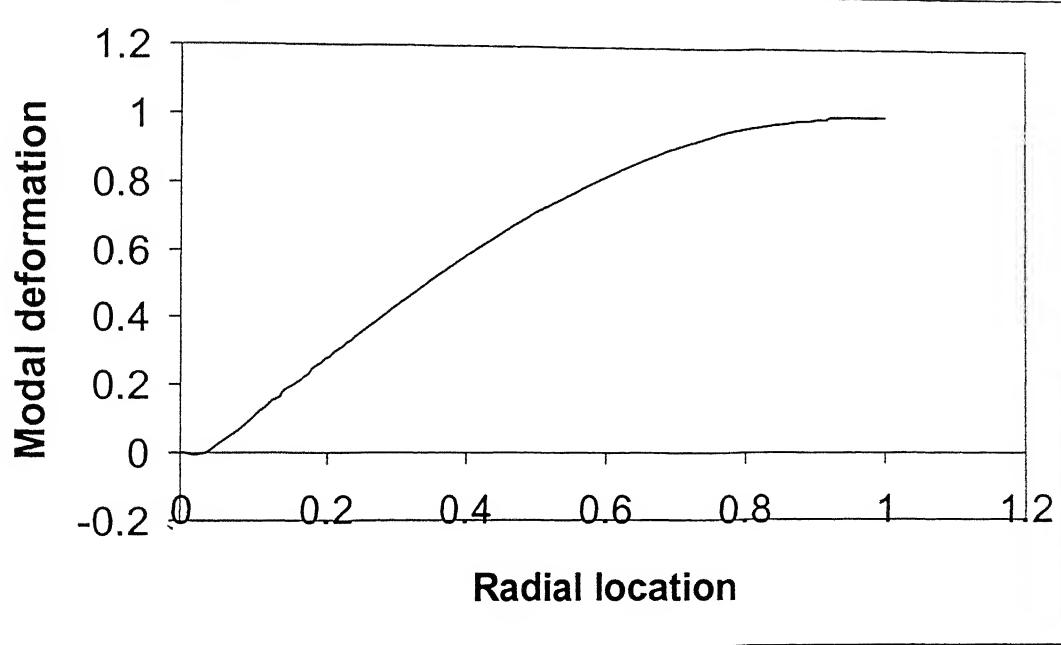
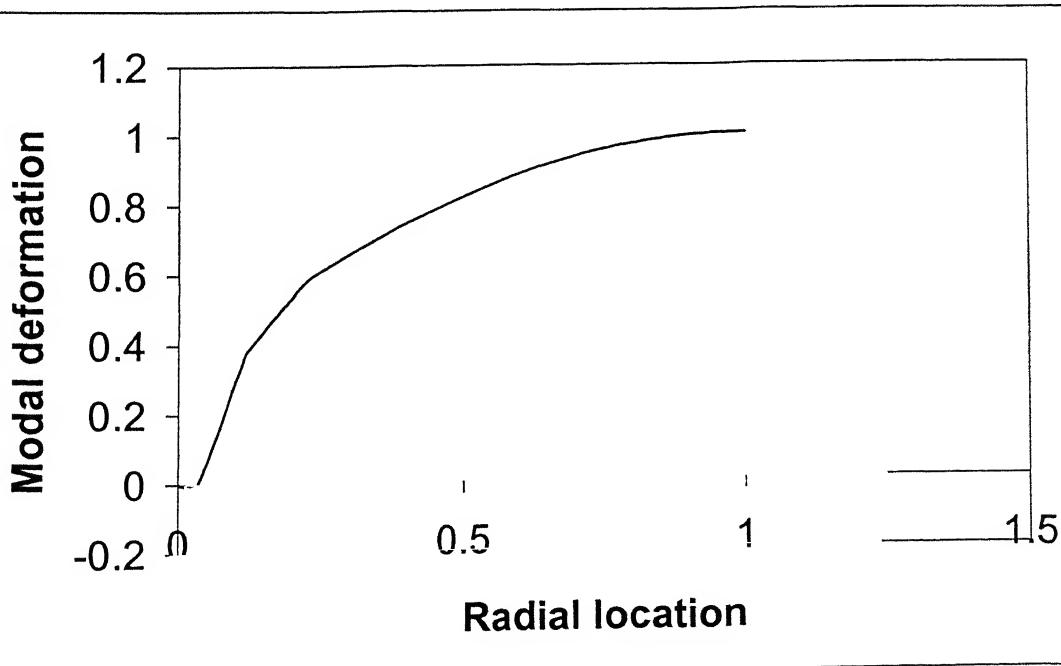


Figure 8.8 Mode shapes in flap (out of plane bending) mode



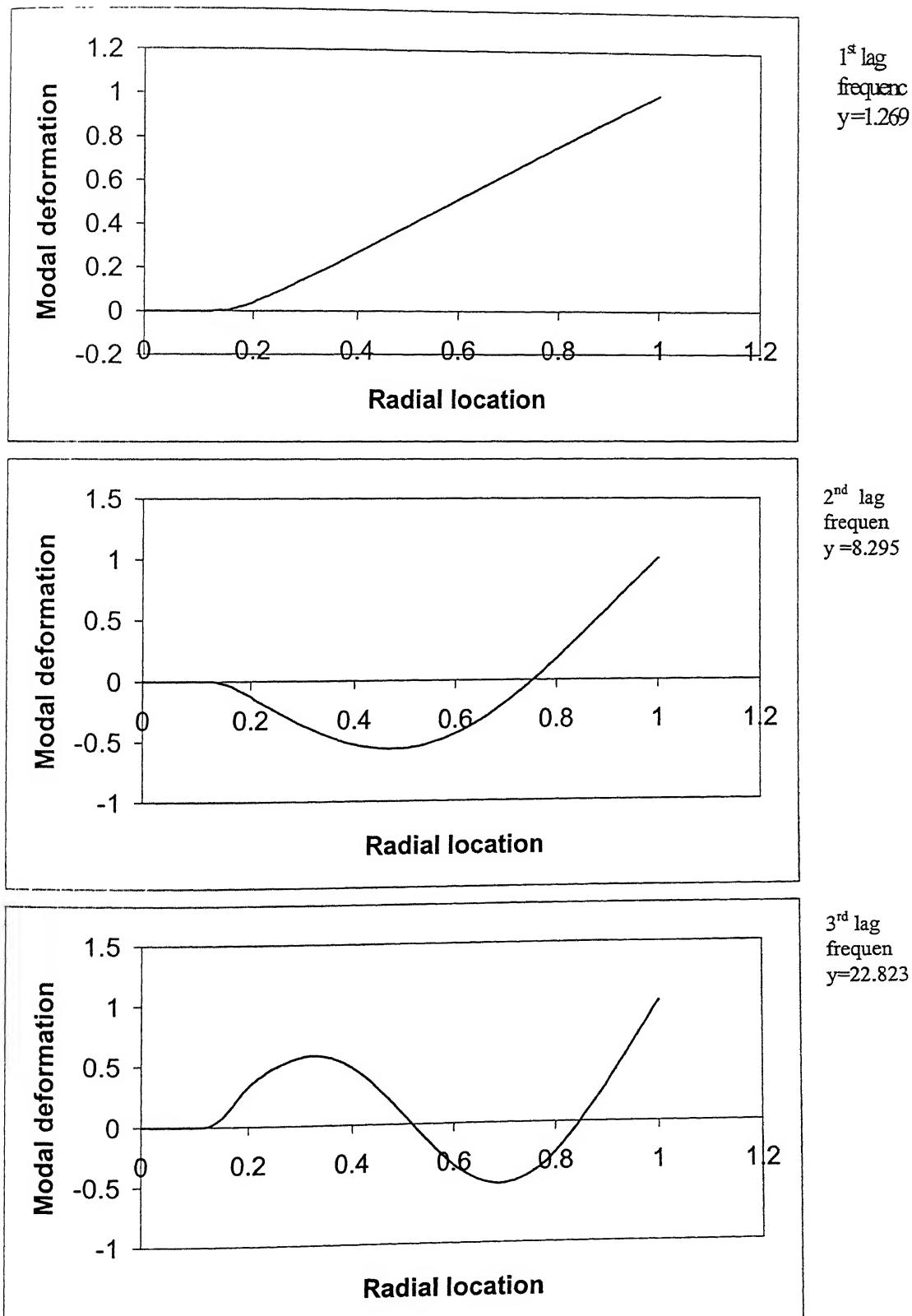
1<sup>st</sup> Axial frequency = 8.333.

Figure 8.9 Mode shape in axial mode

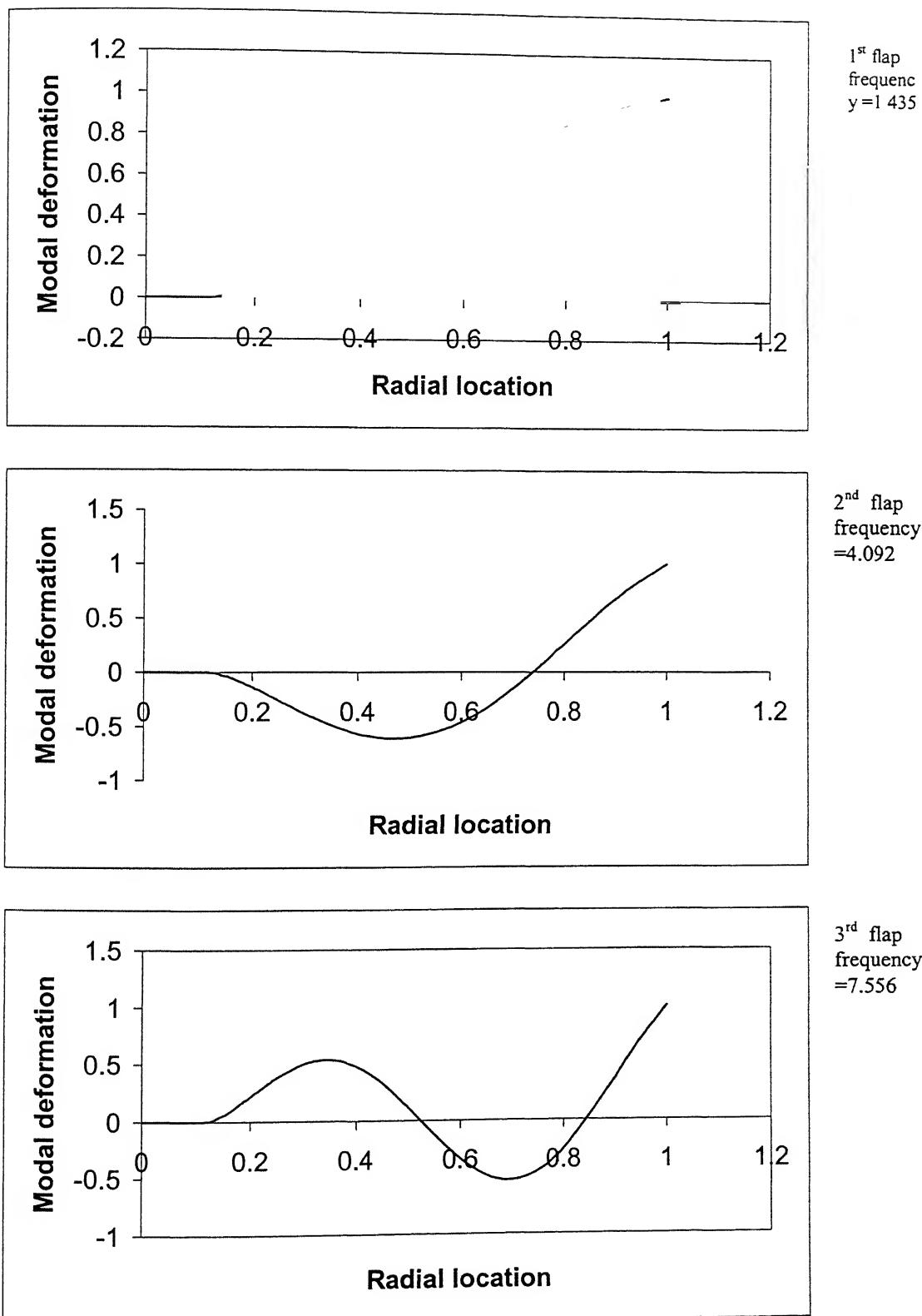


1<sup>st</sup> torsional frequency = 5.626

Figure 8.10 Mode shape in torsional mode

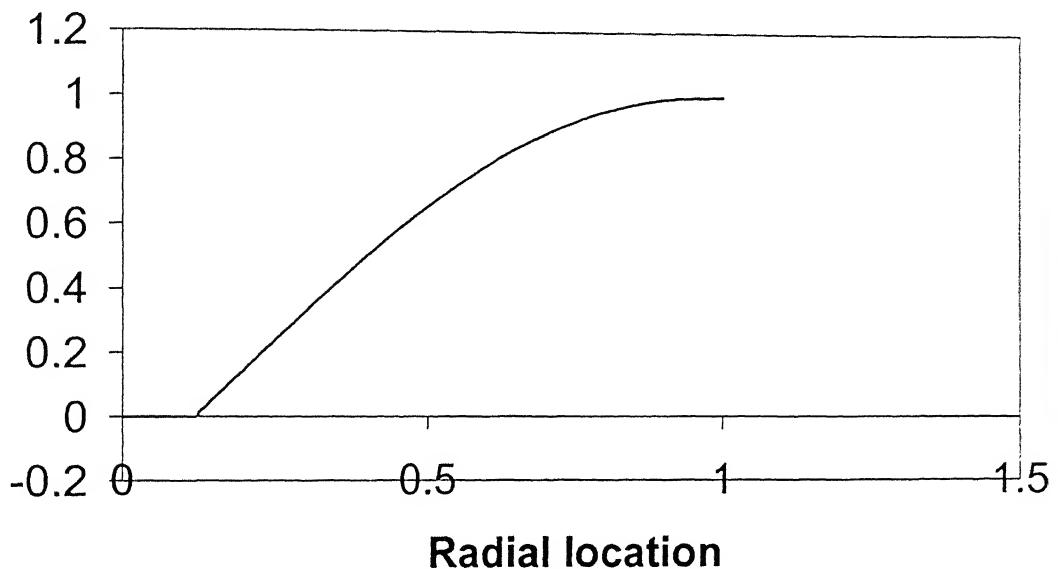


**Figure 8.11** Mode shapes in lead-lag(in-plane bending) mode



**Figure 8.12** Mode shapes in flap(out of plane bending) mode

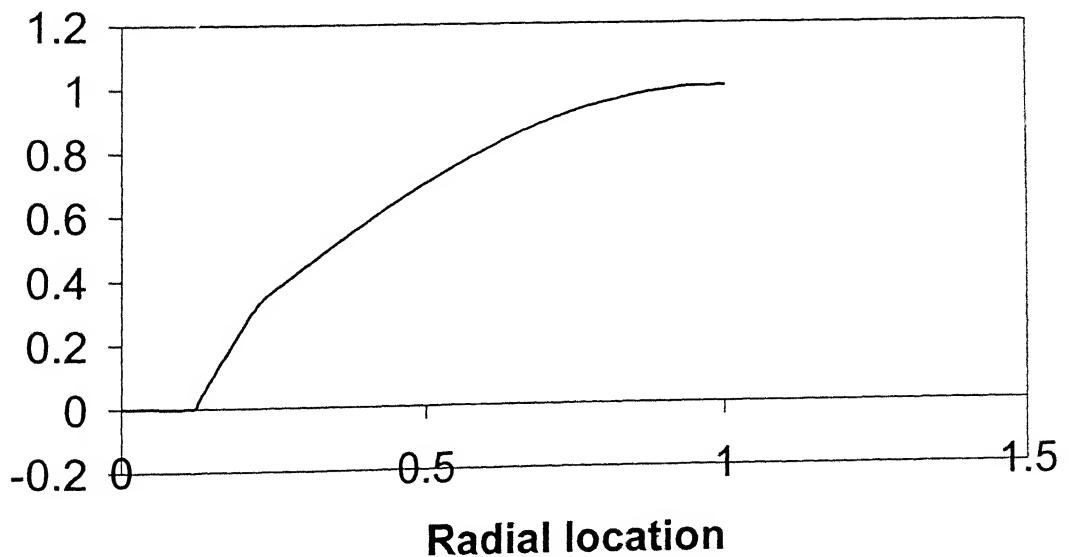
**Modal deformation**



1<sup>st</sup> axial frequency = 9.1577

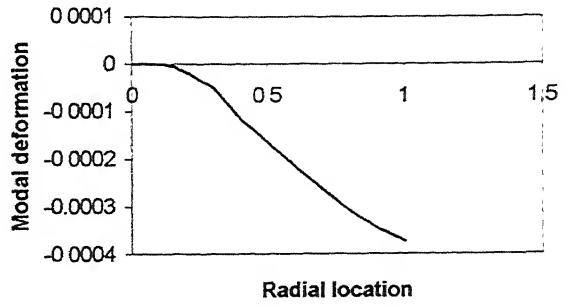
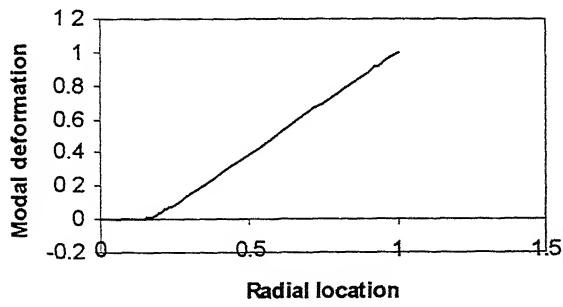
**Figure 8.13** Mode shape in axial mode

**Modal deformation**

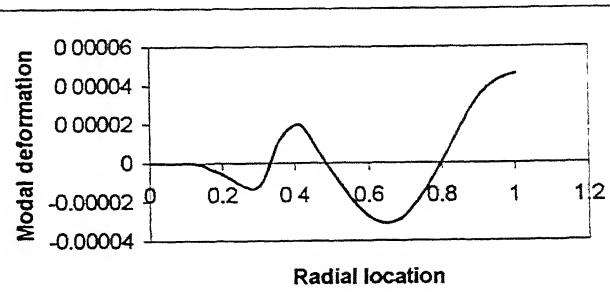
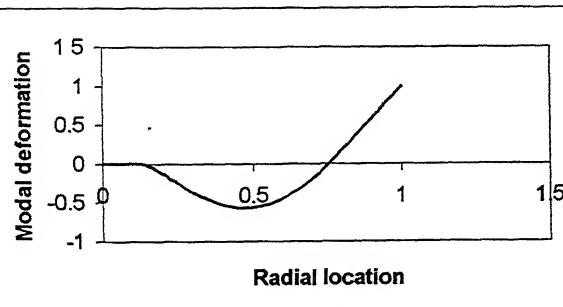


1<sup>st</sup> torsion frequency = 7.508

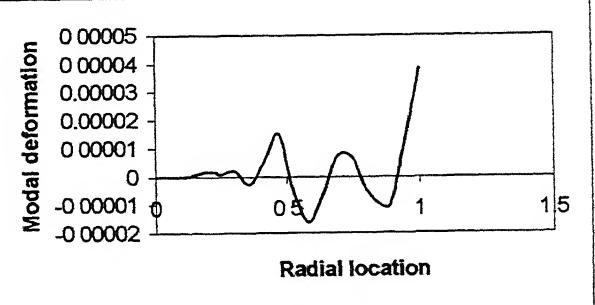
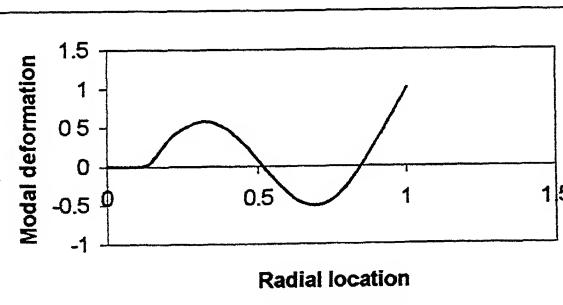
**Figure 8.14** Mode shape in torsional mode



1<sup>st</sup> lag frequency = 1.266

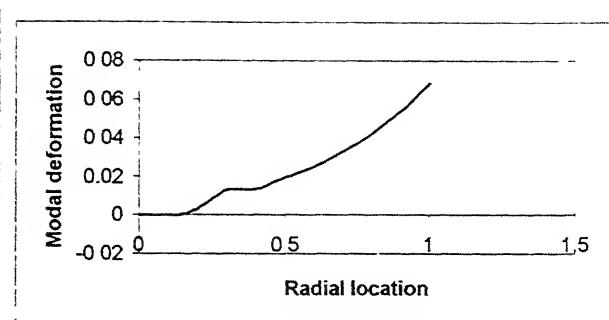
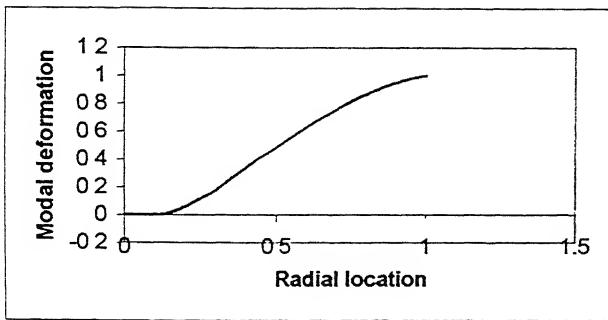


2<sup>nd</sup> lag frequency = 8.224

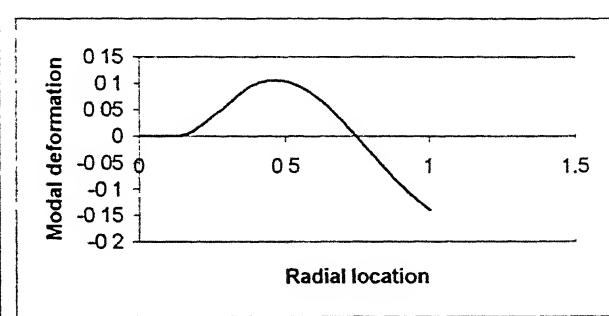
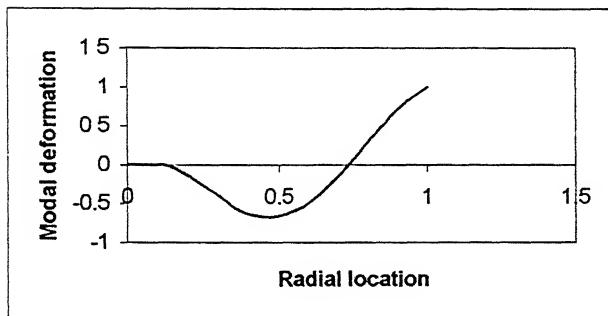


3<sup>rd</sup> lag frequency = 22.632

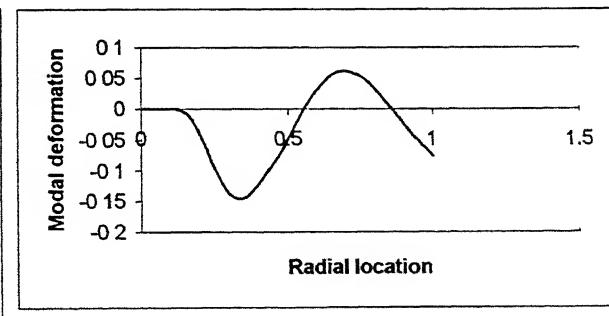
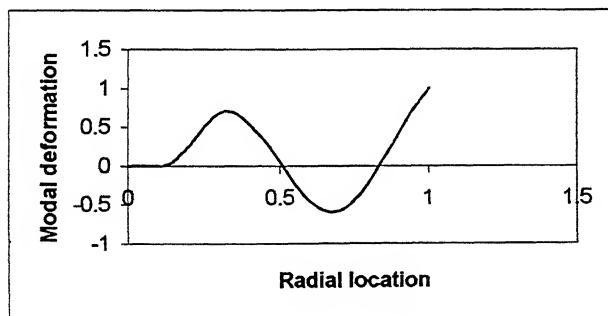
Figure 8.15 Coupled Lead lag mode



1<sup>st</sup> flap frequency = 1.429



2<sup>nd</sup> flap frequency = 3.960



3<sup>rd</sup> flap frequency = 6.623

**Figure 8.16** Coupled flap mode

# Chapter 9

## Concluding Remarks

A general structural dynamic model of a helicopter rotor has been formulated. The model incorporates all the structural complexities present like pre-sweep angle, pre-twist, tip sweep angle, tip anhedral angle etc., Due to practical requirements, in the present formulation, speed of rotation is taken as a variable. The rotor was modeled using beam finite elements having 14 degrees of freedom. The equations of motion are derived using Hamilton's principle. The formulation was validated by comparing the results of the present analysis for a uniform hingeless rotor blade with that of those available in literature. Results have also been generated for practical helicopter rotor blade.

The effect of root offset on the natural frequency and mode shapes of the hingeless rotor blade has been analyzed. The coupling effect of geometric pitch on the natural frequency and mode shapes of the practical rotor blade has also been analyzed.

### 9.1 Scope for the Future Work

Rotor blade aerodynamic model has to be formulated and integrated with the present structural dynamic model. The resulting analytical equations can be used for the study of aero-elastic response and stability of a rotor blade, coupled rotor-fuselage stability and vibration analysis of helicopters.

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## Appendix A

### Local to global coordinate transformation for tip element

The local-to-global coordinate transformation for the swept-tip element can be written in the form

$$\{q^G\} = [\Lambda^L]\{q_t^L\} \quad (A.1)$$

Where the subscript t denotes quantities associated with tip element; the superscripts L and G denote the local and global coordinate system, respectively; q is the vector of element nodal degrees of freedom, the transformation matrixs,  $[\Lambda^L]$ , is derived with constraint that the angular relationship between the swept-tip and the straight portion of the blade at the junction is preserved after deformation [Ref. 3 ]. For the translational degrees of freedom, the transformation is linear, as indicated by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_t^L = [T_e] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_t^G \quad (A.2)$$

where the transformation matrix  $[T_e]$  is given by

$$[T_e] = \begin{bmatrix} \cos \Lambda_s \cos \Lambda_a & -\sin \Lambda_s & \cos \Lambda_s \sin \Lambda_a \\ \sin \Lambda_s \cos \Lambda_a & \cos \Lambda_s & \sin \Lambda_s \sin \Lambda_a \\ -\sin \Lambda_a & 0 & \cos \Lambda_a \end{bmatrix} \quad (A.3)$$

An explicit form of the constraint relations for the rotational degrees of freedom is given by

$$\left\{ \begin{array}{c} \phi \\ -w' \\ v' \end{array} \right\}_t^L = [T_e] \left\{ \begin{array}{c} \phi \\ -w' \\ v' \end{array} \right\}_t^G \quad (A.4)$$

Combining the Eqs. A.2-A.4 and rewriting in the form given in Eq. A.1, the elements of transformation matrix  $\Lambda^L$  are obtained. The elements of the transformation matrix are given below.

$\Lambda^L(1, 1) = \cos \Lambda_s$	$\Lambda^L(1, 5) = \sin \Lambda_s \sin \Lambda_a$
$\Lambda^L(1, 12) = \sin \Lambda_s \cos \Lambda_a$	$\Lambda^L(2, 2) = \cos \Lambda_a$
$\Lambda^L(2, 9) = -\sin \Lambda_a$	$\Lambda^L(3, 3) = \cos \Lambda_s$
$\Lambda^L(3, 7) = \sin \Lambda_s \sin \Lambda_a$	$\Lambda^L(3, 14) = \sin \Lambda_s \cos \Lambda_a$
$\Lambda^L(4, 4) = \cos \Lambda_a$	$\Lambda^L(4, 11) = -\sin \Lambda_a$
$\Lambda^L(5, 5) = \cos \Lambda_a$	$\Lambda^L(5, 12) = -\sin \Lambda_a$
$\Lambda^L(6, 2) = -\sin \Lambda_s \sin \Lambda_a$	$\Lambda^L(6, 6) = \cos \Lambda_s$
$\Lambda^L(6, 9) = -\sin \Lambda_s \cos \Lambda_a$	$\Lambda^L(7, 7) = \cos \Lambda_a$
$\Lambda^L(7, 14) = -\sin \Lambda_a$	$\Lambda^L(8, 4) = -\sin \Lambda_s \sin \Lambda_a$
$\Lambda^L(8, 6) = \cos \Lambda_s$	$\Lambda^L(8, 11) = -\sin \Lambda_s \cos \Lambda_a$
$\Lambda^L(9, 2) = \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(9, 6) = \sin \Lambda_s$
$\Lambda^L(9, 9) = \cos \Lambda_s \cos \Lambda_a$	$\Lambda^L(10, 2) = \frac{1}{2} \cos \Lambda_s \sin \Lambda_a$
$\Lambda^L(10, 4) = \frac{1}{2} \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(10, 6) = \frac{1}{2} \sin \Lambda_s$
$\Lambda^L(10, 8) = \frac{1}{2} \sin \Lambda_s$	$\Lambda^L(10, 10) = \cos \Lambda_s \cos \Lambda_a$
$\Lambda^L(11, 4) = \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(11, 8) = \sin \Lambda_s$
$\Lambda^L(11, 11) = \cos \Lambda_s \cos \Lambda_a$	$\Lambda^L(12, 1) = -\sin \Lambda_s$
$\Lambda^L(12, 5) = \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(12, 12) = \cos \Lambda_s \cos \Lambda_a$
$\Lambda^L(13, 1) = -\frac{1}{2} \sin \Lambda_s$	$\Lambda^L(13, 3) = -\frac{1}{2} \sin \Lambda_s$
$\Lambda^L(13, 5) = \frac{1}{2} \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(13, 7) = \frac{1}{2} \cos \Lambda_s \sin \Lambda_a$
$\Lambda^L(13, 13) = \cos \Lambda_s \cos \Lambda_a$	$\Lambda^L(14, 3) = -\sin \Lambda_s$
$\Lambda^L(14, 7) = \cos \Lambda_s \sin \Lambda_a$	$\Lambda^L(14, 14) = \cos \Lambda_s \cos \Lambda_a$

## Appendix B

### Modification of the Term $[V_{31}^L]$ Associated with Kinetic Energy

Expanding  $\sin(\theta_G + \phi_k)$  and  $\cos(\theta_G + \phi_k)$ , and assuming that  $\phi_k$  is small

$$\sin(\theta_G + \phi_k) \approx \sin \theta_G + \phi_k \cos \theta_G$$

$$\cos(\theta_G + \phi_k) \approx \cos \theta_G - \phi_k \sin \theta_G$$

Substituting the above approximation in  $[V_{31}^L]$ , and rewriting as

$$[V_{31}^L] \approx [V_{31}^L] + [K_{33}^{cf}]\{\phi\}$$

where

$$\begin{aligned} [K_{33}^{cf}] &= \int_0^{l_e} [ \{ (Im_{\zeta\zeta} \cos^2 \theta_G + Im_{\eta\eta} \sin^2 \theta_G ) (\cos^2 \theta_I \cos^2 \Lambda_a \\ &\quad - \cos^2 \theta_I \sin^2 \Lambda_s \sin^2 \Lambda_a ) \\ &\quad - (Im_{\zeta\zeta} \sin^2 \theta_G + Im_{\eta\eta} \cos^2 \theta_G ) \} \{\phi_q\} \{\phi_q\}^T ] dx \\ [V_{31}^L] &= \int_0^{l_e} [ \{ (Im_{\zeta\zeta} - Im_{\eta\eta}) \sin \theta_G \cos \theta_G \cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a \\ &\quad - Im_{\eta\zeta} (\sin^2 \theta_G - \cos^2 \theta_G) \cos^2 \theta_I \cos \Lambda_s \cos \Lambda_a \\ &\quad + (Im_{\zeta\zeta} \sin \theta_G + Im_{\eta\eta} \cos \theta_G) \sin \theta_I \cos \Lambda_a \} \{\phi_q\} ] dx \end{aligned}$$

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